POLYTECHNIQUE MONTRÉAL

TECHNOLOGICAL UNIVERSITY

QUALIFICATION EXAM – WRITTEN PART
Ph. D. Program in Engineering Physics

Thursday October 15, 2020

Room L-1720(Lassonde Building)

from 9h30 to 13h30

NOTES :

- *No documentation is allowed.*
- *A non-programmable calculator is allowed.*
- *A page of mathematical and physics equation is provided at page 2.*
- *This examination contains 8 questions, 10 pages in total.*
- *Each question is worth 20 points.*
- *Provide solutions to no more than 6 questions of your choice.*
- *Use a different notebook for each question, clearly making each notebook with the corresponding question number.*

ENGLISH VERSION

Department Engineering Physics
Main Building
Téléphone : 514-340-4787
Télécopieur : 514-340-3218
Courriel : info@phys.polymtl.ca

Adresse postale
P.O. Box 6079, Station Centre-ville
Montréal (Québec) Canada H3C 3A7
www.polymtl.ca

2900, boul. Édouard-Montpetit
Campus of the'Université of Montréal
2500, chemin de Polytechnique
Montréal (Québec) Canada H3T 1J4

FORMULAS AND CONVERSION

Constantes

$1\ \text{Å} = 10^{-10}\text{m}$	$1\ \text{eV} = 1.602 \times 10^{-19}$ J	$h = 6.626 \times 10^{-34}$ J s
$c = 2.998 \times 10^{8}$ m/s	$k_B = 1.381 \times 10^{-23}$ J/K	$\epsilon_0 = 8.854 \times 10^{-12}$ C^2/ N m^2
$m_e = 9.109 \times 10^{-31}$ kg	$m_p = 1.672 \times 10^{-27}$ kg	$\|e\| = 1.602 \times 10^{-19}$ C
$N_A = 6.023 \times 10^{23}$ mol^{-1}	$a_0 = 0.5291 \times 10^{-10}$ m	$\mu_0 = 4\pi \times 10^{-7}$ N^2/ A^2

Équations physiques

$$\nabla \cdot \mathbf{D} = \rho_f \ ; \quad \nabla \cdot \mathbf{B} = 0 \ ; \quad \nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \ ; \quad \nabla \times \mathbf{H} = \mathbf{J}_f + \frac{\partial \mathbf{D}}{\partial t} \ ; \quad \mathbf{B} = \frac{\mu_0}{4\pi} \int \frac{\mathbf{J} \times \hat{\mathbf{r}}}{|\mathbf{r}|^2} dV$$

$$f_{\text{FD}}(E) = \frac{1}{e^{(E-\mu)/k_B T} + 1} \quad ; \quad f_{\text{BE}}(E) = \frac{1}{e^{(E-\mu)/k_B T} - 1}$$

$$\frac{N\alpha}{2\varepsilon_0} = \frac{\varepsilon_r - 1}{\varepsilon_r + 2} \quad \text{(relation de Clausius-Mossotti)}$$

Intégrales

$$\int \frac{dx}{e^x + 1} = x - \ln(e^x + 1) \quad ; \quad \int \frac{dx}{e^x - 1} = \ln(e^x - 1) - x$$

$$\int_0^\infty x^n e^{-qx} dx = \frac{n!}{q^{n+1}}, \quad n > -1, q > 0 \quad ; \quad \int_0^\infty e^{-\alpha^2 x^2} dx = \frac{1}{2\alpha}\sqrt{\pi}$$

$$\int_0^\infty x^{2n} e^{-\alpha x^2} dx = \frac{1 \cdot 3 \cdot 5 \cdots (2n-1)}{2^{n+1}\alpha^n}\sqrt{\frac{\pi}{\alpha}} \quad ; \quad \int_0^\infty x^{2n+1} e^{-\alpha x^2} dx = \frac{n!}{2\alpha^{n+1}}, \quad (\alpha > 0)$$

Identités trigonométriques

$\cos(2a) = \cos^2 a - \sin^2 a = 2\cos^2 a - 1$ $\qquad$ $\sin(2a) = 2 \sin a \cos a$

$\sin a \sin b = \frac{1}{2}\cos(a-b) - \frac{1}{2}\cos(a+b)$ $\qquad$ $\cos a \cos b = \frac{1}{2}\cos(a-b) + \frac{1}{2}\cos(a+b)$

$\sin a \cos b = \frac{1}{2}\sin(a+b) + \frac{1}{2}\sin(a-b)$ $\qquad$ $\cos a \sin b = \frac{1}{2}\sin(a+b) - \frac{1}{2}\sin(a-b)$

$\cos x = (e^{ix} + e^{-ix})/2$ $\qquad$ $\sin x = (e^{ix} - e^{-ix})/2i$

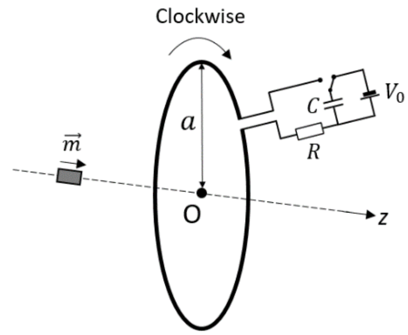
Autres

$$c^2 = a^2 + b^2 - 2ab\cos\gamma \ \text{(loi des cosinus)} \quad ; \quad n! \approx n^n e^{-n}\sqrt{2\pi n} \ \text{(Approximation de Stirling)}$$

$$\nabla^2 \mathbf{A} = \nabla(\nabla \mathbf{A}) - \nabla \times \nabla \times \mathbf{A} \ \text{(Identité)} \quad ; \quad \sum_{n=0}^{\infty} x^n \approx \frac{1}{1-x}, |x| < 1 \ \text{(Série)}$$

QUESTION 1 : **ELECTROMAGNETISM**

Magnetic cannon



Consider the following, simple model of a magnetic cannon (see figure) designed to accelerate a magnetized projectile. The cannon is made of a conductive loop of radius a, located in the $z = 0$ plane, through which flows a current $I(t)$ generated by the discharge of a RC circuit (the inductance of the loop is negligible here). The z axis coincides with the axis of the loop.

The projectile has a magnetic moment $\vec{m} = m\hat{z}$ ($m > 0$) and is initially at rest on the z axis. It is then accelerated towards the right and stays on the z axis at all times. The projectile is moving through air.

We suppose that the characteristic time τ of the RC circuit is extremely short, so that its discharge can be modeled by a brief current pulse in the temporal domain.

(a) (**2 pts)** In what direction does the current have to flow through the loop (clockwise or counterclockwise, see figure) to increase the velocity of the projectile if it is located in the region : i) $z < 0$? ii) $z > 0$?

(b) **(8 pts)** Find the expression of the magnetic field $\vec{B}(z,t)$ generated by the conductive loop on its axis.

(c) **(6 pts)** Find the expression of the force $\vec{F}(z,t)$ exerted on the projectile.

(d) **(4 pts)** Sketch a graph of the force $\vec{F}(z,t)$ against the position of the projectile at a given time. Describe how you would proceed to maximize the final velocity of the projectile, given that it is initially at rest.

Biot-Savart's law

$$\vec{B}(\vec{r}) = \frac{\mu_0}{4\pi} \int \frac{Id\vec{l}' \times (\vec{r} - \vec{r}')}{|\vec{r} - \vec{r}'|^3}$$

Magnetic energy of a magnetic dipole

$$U = -\vec{m} \cdot \vec{B}$$

Force on a magnetic dipole

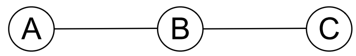
$$\vec{F} = (\vec{m} \cdot \nabla)\vec{B}$$

Discharge of a RC circuit

$$I(t) = I_0 e^{-t/\tau}, \qquad \tau = RC$$

***QUESTION* 2** : **QUANTUM MECHANICS**

An electron is found in a linear molecule made up of three equidistant atoms. We will use $|A\rangle$, $|B\rangle$ and $|C\rangle$ to describe the orthonormal states of the electron that correspond to localized states on respectively, on atom A, B and C.



If we neglect the possibility that the electron jumps from one atom to another, its energy is described by the Hamiltonian $\hat{H}_0$ such as

$$\hat{H}_0 = \begin{pmatrix} E_0 & 0 & 0 \\ 0 & E_0 & 0 \\ 0 & 0 & E_0 \end{pmatrix}$$

where the eigenstates are the three states $|A\rangle$, $|B\rangle$ and $|C\rangle$ with the same eigenvalue E_0. We then consider the perturbation

$$\hat{H}' = \begin{pmatrix} 0 & a & 0 \\ a & 0 & a \\ 0 & a & 0 \end{pmatrix}$$

(a) **(8 pts)** If *a* is a real positive constant, find the eigenvalues and eigenstates of the perturbed system such as $\hat{H} = \hat{H}_0 + \hat{H}'$. Give the stationary wavefunctions of states $|A\rangle$, $|B\rangle$ and $|C\rangle$ in terms of eigenstates of the perturbed system. (8 points)

(b) **(6 pts)** If the electron is in the state $|A\rangle$ at the time $t = 0$. Give the expression for the probability of finding the electron on atom A when $t > 0$. Express your answer in terms of cos and sin functions.

(c) **(6 pts)** Is, for a given time $t \neq 0$, the electron can be found totally localized on atom A ? If yes, determine that time, if not, say why it is not possible.

QUESTION 3 : **STATISTICAL PHYSICS**

We consider the energy levels of a set of diatomic molecules and consider 4 degrees of freedom under the assumption that the total energy can be written as:

$$E_{totale} = E_{translation} + E_{électronic} + E_{rotationnal} + E_{vibrationnal} = E_t + E_e + E_r + E_v$$

(a) **(4 pts)** Show that the partition function can be written as:

$$Z_{totale} = Z_t Z_e Z_r Z_v$$

(b) **(4 pts)** Knowing that the harmonic oscillator energy is given by $E_v = \left(n + \frac{1}{2}\right)\hbar\omega$, calculate the distribution of vibrational states according to the Maxwell-Boltzmann law.

(c) **(4 pts)** Knowing that the rotational energy is $2\ell + 1$ degenerate and can be written as follows : $E_r = \frac{\hbar}{\mu R^2}\big(\ell(\ell+1)\big)$ where μ is the reduced mass of the molecule and R is the interatomic distance, give the expression of the partition function.

(d) **(4 pts)** Assuming that $\frac{\hbar}{\mu R^2} \ll k_B T$, we can replace the summation in the expression of the partition function by an integral. Show that the partition function for rotational energies can then be approximated by $\frac{k_B T \mu R^2}{\hbar}$.

(e) **(4 pts)** The initial assumption that the rotational and vibrational energies are independent is only valid for vibrational levels near the ground state (beyond that, the potential well anharmonicity tends to change the average interatomic distance, which is considered constant in the model used to calculate rotational energy). At room temperature and knowing that vibrational state transitions are of the order of 0.1 eV, is this assumption generally respected?

QUESTION 4 : CLASSICAL MECHANICS

Coupled torsion spring system

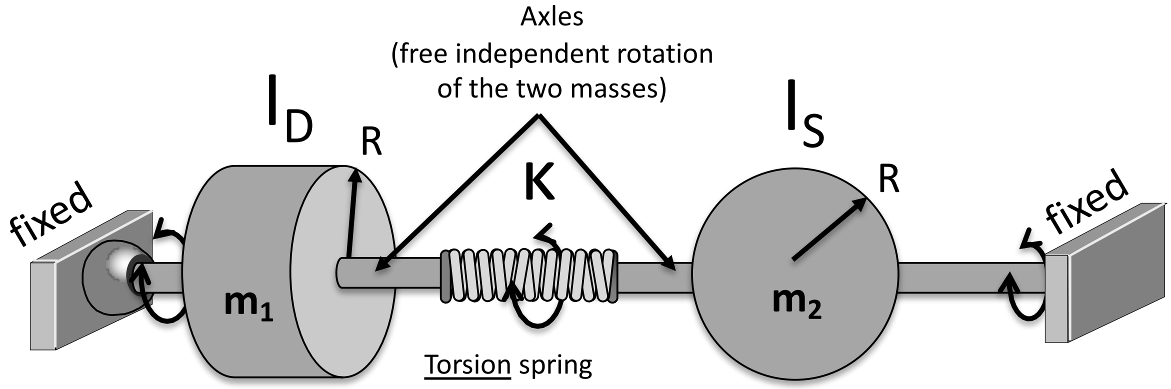


Figure 1 : *Schematics of the system under study*

A solid disc and a solid sphere of respective mass m_1 et m_2 and or identical radius R are fixed on two massless half axles and held together by a torsion spring of constant K. The axle can spin freely around its axis.

At time t = 0, we hold the sphere in place while we rotate the cylinder 90 degrees around the axis. We then let the system loose, starting at rest.

(a) **(4 pts)** Specify a set of generalized coordinates to study this problem. Describe precisely what the variables mean and their reference point.

(b) **(8 pts)** Write the hamiltonian of the system.

(c) **(4 pts)** Obtain Hamilton's equations describing the system.

(d) **(4 pts)** Knowing that we can distribute a mass M in both objects, such that $M = m_1 + m_2$, in which proportions must m_1 et m_2 be distributed to maximize the oscillation frequency of the system?

Useful information :

- *Neglect any friction;*
- *The moment of inertia of a solid disk of mass m around its central axis is $I_D = ½*mR^2$. The moment of inertial of a solid sphere around its principal axis is $I_S = 2/5*mR^2$.*
- *A torsion spring generates a tork that is directly proportional to its torsion angle ($M = -K\vartheta$).*
- *The spring cannot be deformed in x or y, only angularly.*

Note : Any other approach than Hamiltonian mechanics is valid, albeit with a 2 point deduction (out of 20).

QUESTION 5 : **GEOMETRICAL OPTICS**

1. **(5 pts) Refraction** : When the Sun is setting, its rays must pass through Earth's atmosphere before reaching our eyes. Using a sketch and a brief explanation, and using your knowledge of refraction, argue whether or not, the apparent position of the Sun when near the horizon provides an accurate determination of the duration of the day. In other words, when we look at a sunset, is the Sun on, below or above the horizon.

2. **Bessel's method**: To evaluate the focal distance of a lens, f, Bessel showed that it may be placed between an object and a screen as long as the object and the screen are separated by a long distance, L, with $L > 4f$.

   (a) **(5 pts)** Show that there exists two positions, $x_1$and x_2, for the lens for which the image of the object on the screen is sharp;

   (b) **(5 pts)** Let D be the distance between $x_1$and x_2: the two lens positions providing a sharp image on the screen. Show that the focal lens of the lens obeys: $f = \frac{L^2-D^2}{4L}$;

   (c) **(5 pts)** Propose, sketch and briefly explain another method for determination of the focal length of the lens.

QUESTION 6 : **WAVES OPTICS**

Recent measurements using state-of-the-art optical interferometers have shown that gravitational waves from black holes can be detected. The scheme uses two ultra-stable Fabry-Perot cavities, FP1 and FP2, embedded within the two arms of a very long ultra-stable Michelson interferometer. This system is shown in Fig. 1.

An ultra-stable laser is used as the source for the gravitational wave detector. The output is detected by a perfect photodetector, PD1, that can be considered free of any sources of background noise or photonic noise.

In this problem you can make the following assumptions:

- A gravitational wave arriving at the interferometer affects only one arm of the interferometer while leaving the other arm unaffected.
- The wave causes a gravitational strain, ε, defined as $\delta l/L$, in one arm, where δl is the change in the length, L, of the interferometer.
- The interference at the photodetector is destructive before a gravitational wave hits it.
- Light in both FP1 and FP2 undergo N round trips (with N = 101) before exiting and interfering at the photodiode: one round trip is from mirror M1 to M2 and back to M1 (FP1), and from mirror M3 to M4 and back to M3 (FP2).
- The wavelength of the laser is 532nm.
- The cavity lengths of FP1 and FP2 are identical and equal to 5000 m each.
- The power, P_0, of the laser source is 121W.
- The current response of the photodiode is I = 0.8 x δP where δP is the detected light power.

(a) **(15 pts)** Calculate the change in the current detected on the photodetector if the gravitational strain in one arm is $\varepsilon = 0.9 \times 10^{-21}$.

(b) (**5 pts**) Calculate the detected number of photons/second.

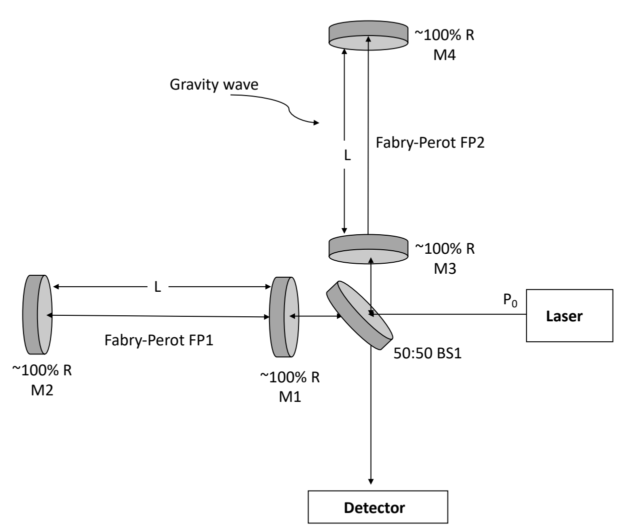


Figure 1. Schematic of two Fabry-Perot interferometers embedded in a Michelson Interferometer for gravitational wave detection (R : reflectivity, M : mirrors, BS : beam splitter).

QUESTION 7 : SOLID STATE PHYSICS 1

Consider normal modes of vibration of a linear chain of identical atoms of mass M in the harmonic approximation. Assume that coupling between atoms exist only for nearest neighbor atoms. These couplings are represented by single equal springs with force constant K. Normal modes have wave vector **k** and frequency w. Also assume that the chain has N atoms and length $L = Na$.

(a) **(5 pts)** Write the equation of motion for displacements **u** of normal modes and obtain the normal mode frequencies $w(k)$ as function of the magnitude of the wavevector $|\mathbf{k}| = k$.

(b) **(3 pts)** Longitudinal normal modes have $\mathbf{u} \parallel \mathbf{k}$ and transverse modes have $\mathbf{u} \perp \mathbf{k}$. How many longitudinal modes and transverse modes exist for each value of wave vector. Why are the modes with wave vectors **k** and -**k** degenerate?

(c) **(7 pts)** Show that for long wavelengths the equation of motion reduces to a continuum elastic wave equation. What is the speed of sound?

(d) **(5 pts)** Use the cyclic boundary conditions to obtain the density of states of modes. How many modes are in the first Brillouin zone of the linear chain?

QUESTION 8 : **SOLID STATE PHYSICS 2**

A semiconducting sample displays a mobile carrier density of 2×10^{16} cm^{-3} and a mobility coefficient $\mu = 80$ $cm^2/V \cdot s$.

(a) **(4 pts)** Calculate the electrical conductivity of this semiconducting sample.

(b) **(8 pts)** Consider thermal equilibrium at room temperature. In this state, the carriers are continuously being trapped into immobile sites and then are thermally reionized into mobile states. If the average lifetime in a mobile state is 10^{-5} seconds, what is the average distance a carrier diffuses between successive trappings?

(c) **(8 pts)** If the charge carriers have an effective mass equal to 0.1 times the mass of a free electron, what is the average time between successive scatterings?

Note:
At room temperature, $k_B T/e = 25$ meV.