POLYTECHNIQUE MONTRÉAL
TECHNOLOGICAL UNIVERSITY

QUALIFICATION EXAM – WRITTEN PART
Ph. D. Program in Engineering Physics

<u>Thursday June 13, 2019</u>

Room B-418 (Main Building)

from 9h30 to 13h30

<u>NOTES</u> :

- *No documentation is allowed.*
- *A non-programmable calculator is allowed.*
- *This examination contains 10 questions, 11 pages in total.*
- *Each question is worth 20 points.*
- *Provide solutions to no more than 6 questions of your choice.*
- *Use a different notebook for each question, clearly making each notebook with the corresponding question number.*

ENGLISH VERSION

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$$\sum_{n=0}^{\infty} x^n = \frac{1}{1-x}, \quad |x|<1$$

PHYSICAL CONSTANTS

$e = 1.602 \times 10^{-19}$ C
$m_e = 9{,}109 \times 10^{-31}$ Kg
$\hbar = 1.055 \times 10 \times 10^{-34}$ J·s
$k_B = 1.381 \times 10^{-23}$ J/K
$\varepsilon_0 = 8.854 \times 10^{-12}$ F/M
$\mu_0 = 4\pi \times 10^{-7}$ N/A^2

PHYSICS EQUATIONS :

$$\nabla \cdot \mathbf{D} = \rho_f$$

$$\nabla \cdot \mathbf{B} = 0$$

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$$

$$\nabla \times \mathbf{H} = \mathbf{J}_f + \frac{\partial \mathbf{D}}{\partial t}$$

$$\mathbf{B} = \frac{\mu_0}{4\pi} \int \frac{\mathbf{J} \times \hat{\mathbf{r}}}{|\mathbf{r}|^2} dV$$

$$f_{FD}(E) = \frac{1}{e^{(E-\mu)/k_B T} + 1}$$

$$f_{BE}(E) = \frac{1}{e^{(E-\mu)/k_B T} - 1}$$

Clausius-Mossotti relation

$$\frac{n\alpha}{3\varepsilon_0} = \frac{\varepsilon_r - 1}{\varepsilon_r + 2}$$

FORMULAS AND CONVERSION

$$\int_{-\infty}^{\infty} \Psi_v^* \Psi_v dx = \alpha \int_{-\infty}^{\infty} \Psi_v^* \Psi_v dy = \alpha \int_{-\infty}^{\infty} H_v^2(y) e^{-y^2} dy = \alpha \pi^{1/2} 2^v v!$$

$v! = v(v-1)(v-2)\ldots$

1 u = 931,494 MeV/c^2

MATHEMATICS EQUATIONS

Integrals

$$\int_0^{\infty} \sqrt{x}\, e^{-x}\, dx = \frac{\sqrt{\pi}}{2}$$

$$\int_0^{\infty} e^{-ax^2}\, dx = \frac{1}{2}\sqrt{\frac{\pi}{a}}$$

$$\int_0^{\infty} x^2 e^{-ax^2} dx = \frac{\sqrt{\pi}}{4a^{3/2}}$$

Law of cosines

$$c^2 = a^2 + b^2 - 2ab \cos \gamma$$

Stirling's approximation

$$n! \approx n^n e^{-n} \sqrt{2\pi n}$$

Identity

$$\nabla^2 \mathbf{A} = \nabla(\nabla \mathbf{A}) - \nabla \times \nabla \times \mathbf{A}$$

Trigonometric identities

$$\sin(\alpha \pm \beta) = \sin\alpha \cos\beta \pm \cos\alpha \sin\beta$$
$$\cos(\alpha \pm \beta) = \cos\alpha \cos\beta \mp \sin\alpha \sin\beta$$

$$\sin 2\theta = 2 \sin\theta \cos\theta$$
$$\cos 2\theta = \cos^2\theta - \sin^2\theta$$

$$\cos\alpha \cos\beta = \frac{1}{2}\left(\cos(\alpha+\beta) + \cos(\alpha-\beta)\right)$$

$$\sin\alpha \sin\beta = \frac{1}{2}\left(\cos(\alpha-\beta) - \cos(\alpha+\beta)\right)$$

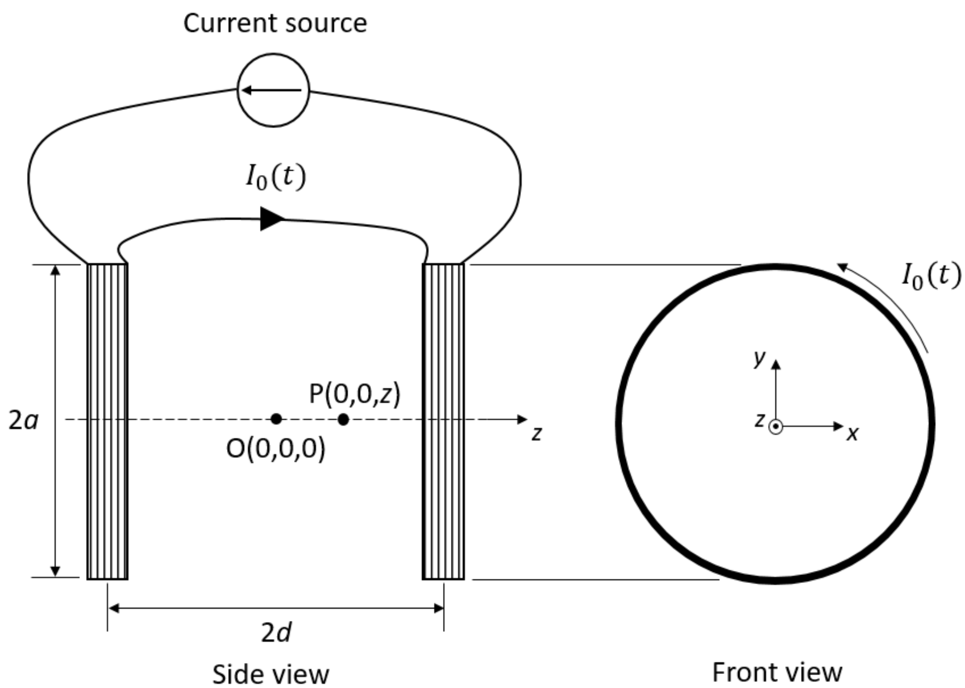
$$\sin\alpha \cos\beta = \frac{1}{2}\left(\sin(\alpha+\beta) + \sin(\alpha-\beta)\right)$$

QUESTION 1 : **ELECTROMAGNETISM**

Generation of a uniform magnetic field by Helmholtz coils

Consider two identical N-turn Helmholtz coils (circular coils), each of radius a. The coils are connected in series and fed by an alternative current source of amplitude I_0. The coils are parallel and aligned with the z axis (see figure). The surrounding medium is air ($\varepsilon_r = \mu_r = 1$). The thickness of the coils in the z direction is negligible.

Helmholtz coils are designed to produce a magnetic field which is (almost) uniform in the surroundings of point $O(0,0,0)$. This point O is located on the axis of the coils and is equidistant from each coil. By placing a sample at this point, it is then possible to study its magnetic response.



a) **(10 points)** Calculate the amplitude of the magnetic field $\vec{B}$ produced by these two Helmholtz coils at an arbitrary point $P(0,0,z)$ located on their axis.

b) **(5 points)** Show that one must set $2d = a$ to maximize the uniformity of the magnetic field in the surroundings of point O. Calculate the amplitude of the magnetic field at point O in this configuration.

   Hint : try minimizing the spatial variation of $\vec{B}$ around point O by making a Taylor expansion around this point :

$$f(x_0 + h) = f(x_0) + f'(x_0)h + \frac{1}{2}f''(x_0)h^2 + O(h^3),$$

   and by setting derivatives equal to zero up to 2nd order.

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From now on, assume that $2d = a$. Let $a = 10$ cm, $N = 10$ and $I_0 = 2$ A.

c) **(5 points)** Give an estimated value of the total inductance (self-inductance) of the two Helmholtz coils. For that purpose, you may assume that the value of the magnetic field on the surface of each coil is uniform and equal to the value of the field at the centre of the coil.

***QUESTION* 2** : **QUANTUM MECHANICS**

a) **(4 points)** Show that in one dimension and a given energy, the wave functions of the time independent Schrödinger equation are either even or odd when the potential is even (V (x) = V (-x)).

b) **(6 points)** Compute the wave functions and associated energies for a particle in an infinite potential well of width a, centered at 0. Is this consistent with the answer to question a?

c) **(2 points)** Compute the average position for the ground state n = 1

d) **(6 points)** Compute the average position as a function of time for a state which, at time t = 0, is a linear combination with equal (real) coefficients of states n = 1 and n = 2.

e) **(2 points)** Find a relationship linking the frequency of the average position and the size of the well.

QUESTION 3 : **STATISTICAL PHYSICS**

Quantum dots

Quantum dots are used in optoelectronics and considered promising technologies for building quantum computers. These dots are nanometric size semiconductor crystals grown on a substrate. They confine the electrons in three dimensions at a nanometric scale. We can analyze their thermodynamic properties using a simplified model (see Figure 1) where at a temperature $T = 0$ °K, one can assume that two electrons are found in the crystal valence band of energy $\varepsilon_\text{v} = -0.2$ eV (ε_v is energy of the valence band). At this temperature, the conduction band of minimum energy $\varepsilon_\text{c} = 0.1$ eV is not occupied.

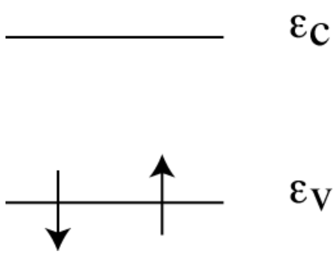


Figure 1. Simplified electronic configuration of a quantum dot.

The questions that needs to be answered are the following.

a) **(5 points)** Find a relation that gives the number of electrons in each band (valence and conduction) as a function of the temperature T and the chemical potential μ.

b) **(5 points)** Show, using the fact that the number of electrons in the crystal remains constant, that the chemical potential μ is independent of the temperature.

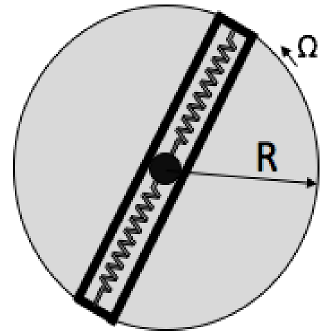
c) **(5 points)** Show that the average number of electrons in the conduction band depends only on $(\varepsilon_\text{v} - \varepsilon_\text{c})$ and T and evaluate the electron population in the conduction band at T=300 °K.

d) **(5 points)** Evaluate the average energy of the electrons in quantum dots when the temperature is $T = 1000$ °K.

The Boltzmann constant is given by $k_B = 8.617 \times 10^{-5}$ eV/ °K.

QUESTION 4 : **CLASSICAL MECHANICS**

Rotating spring-mass system



A spring mass system is made of a marble of mass *m* enclosed in a rigid tube. Two springs of identical spring constants *k* and of negligible lengths at rest are attached to the marble (see figure). The tube is fixed to a horizontal rotating plate (much like a record player). Initially, the marble is at rest and slightly displaced from its position of equilibrium. The marble is then set free and the oscillations are measured.

A) **(5 points)** Determine, as a function of the parameters of the system, the period of oscillation of the marble for the particular case where the plate does not rotate.

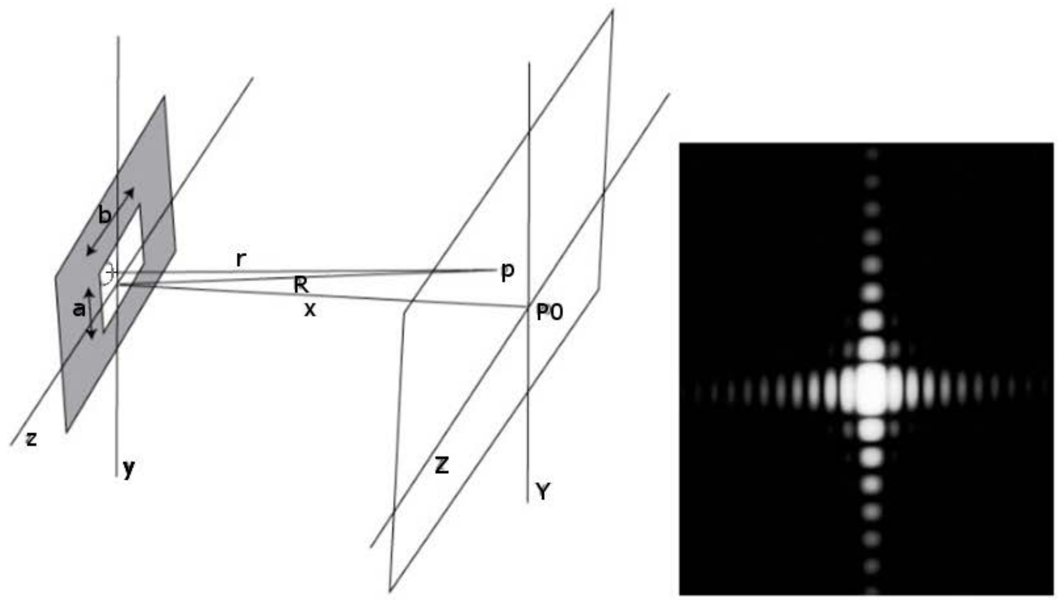
B) **(10 points)** The plate is now set in motion at a constant angular velocity Ω. Determine the kinetic and potential energy of the marble for any radial position r taking the disk center as the origin of the coordinate system. Obtain the lagrangian of the system from these values.

C) **(3 points)** Determine the new period of oscillation of the marble as a function of the parameters of the system, taking into account plate rotation this time.

D) **(2 points)** Suppose that the marble has a mass m = 100 g and a spring of constant k = 1 N/m, calculate the angular velocity required for the system to stop oscillating.

QUESTION 5 : **OPTICS I**

A laser beam is incident from the left (pointed in the $+\hat{x}$ direction) upon a rectangular aperture with length a in the y-direction and length b in the z-direction, where $b > a$ (see Figure). The laser is monochromatic with a wavelength λ and its intensity is uniform across the opening. Light passing through the aperture is collected on a screen at a very large distance $x \gg (a, b, \lambda)$ away from both the aperture and the laser. Coordinates on the distant screen are denoted by their physical coordinates (Y, Z), or by the angles ($\theta_Y \approx Y/x, \theta_Z \approx Z/x$) measured with respect to the $\hat{x}$ axis.



(a) **(4 points)** The intensity pattern $I(\theta_Y, \theta_Z)$ as measured on the distant screen is shown in the Figure on the right. Which directions in this figure correspond to the Y axis, and which correspond to the Z axis? (example answer: the vertical direction corresponds to the Y axis) Explain your reasoning.

(b) **(4 points)** How does the angular distribution of intensity $I(\theta_Y, \theta_Z)$ change if the distance between the screen and the aperture is doubled? Explain your answer.

(c) **(4 points)** How would the measured intensity distribution $I(\theta_Y, \theta_Z)$ change if the wavelength of the laser light λ is doubled? Briefly explain your answer.

(d) **(8 points)** Calculate the intensity distribution $I(\theta_Y, \theta_Z)$ in terms of the angles θ_Y, θ_Z and normalized to the intensity $I(0)$ at the center of the screen. *Hint: Do not worry about the constant amplitude in front of the electric field or intensity until the end of the derivation.*

QUESTION 6 : **OPTICS 2**

Optics : Measurement of a thermo-optic coefficient

The experimental setup shown in Fig. 1 allows the measurement of the thermo-optic coefficient of liquids. The setup comprises a source (S) of diffuse light, a triangular prism, a converging lens (L) and an observation screen placed at its focal length. The liquid to be tested is contained in a cavity (shown in gray) such that the diagonal surface of the prism is in contact with the liquid. The prism is made of a high-index glass with $n = 1{,}9$. The cavity is filled with liquid water having a refractive index $n = 1{,}333$ at 20 °C.

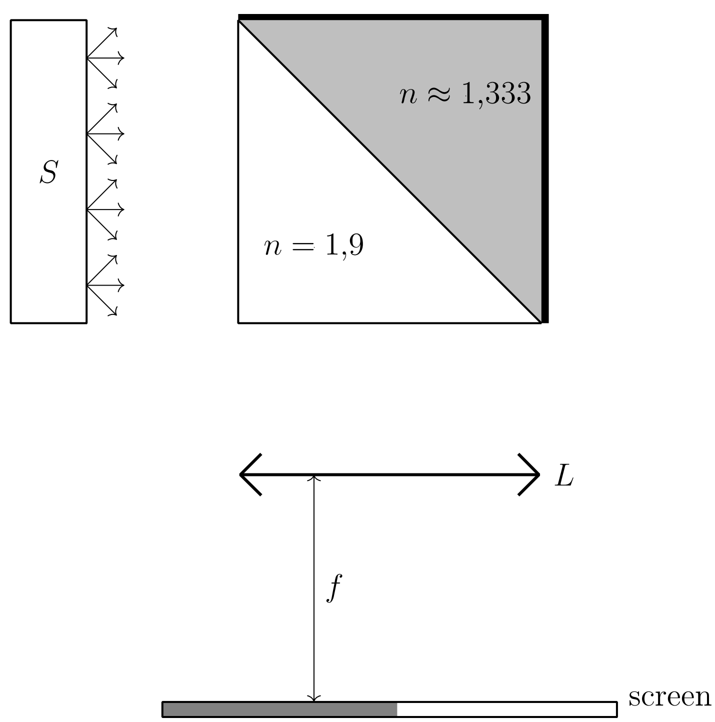


Figure 1 : Measurement setup of the thermo-optic coefficient.

1. **(6 points)** An observation of the screen reveals a bright region and a darker region. Explain the origin of this contrast. Clarify your reasoning and include calculations if required.

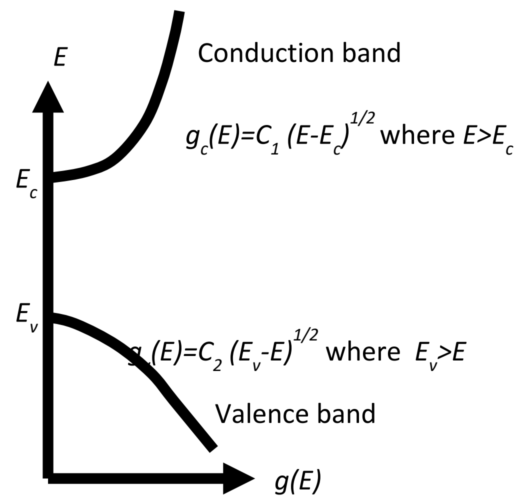
2. **(7 points)** The temperature of the liquid is now modified. Describe how the observation on the screen changes.

3. **(7 points)** Give a procedure allowing the determination of the thermo-optic coefficient $C = \frac{1}{n}\frac{dn}{dT}$.

QUESTION 7 **: SOLID STATE PHYSICS I**

Consider a weakly doped semiconductor. Let the densities of states of the conduction band and valence given by:

$g_c(E)=C_1\,(E-E_c)^{1/2}$ for the conduction band (where $E>E_c$)
$g_v(E)=C_2\,(E_v-E)^{1/2}$ for the valence band (where $E_v>E$)



where $C_1$and C_2 are constants depending only on the characteristics of the semiconductor and the temperature T. E_c and E_v are respectively the minimum of the conduction band and the maximum of the valence band. The Fermi level is denoted by E_F. Make the assumption that
$E_c-E_F >> K_BT$ and $E_F-E_v >> K_BT$. (Note : $k_BT= 0.025\ eV$ at $T=300K$)

a) **(10 points)** Determine the expression of the concentration of electrons n in the conduction band as a function of k_B, T, C_1, E_c, E_F and the expression of the concentration of the holes p in the valence band as a function of k_B, T, C_2, E_v, E_F. There will be a definite integral without dimension in these two expressions that you do not have to evaluate.

b) **(2 points)** Show that the product np is proportional to $exp(-E_g/k_BT)$, where E_g is the gap $E_c–E_v$

c) **(8 points)** Silicon is doped with a low concentration of donors $N_D= 10^{15}\ cm^{-3}$.

   i) Using appropriate assumptions, estimate the electron and hole concentrations at $T = 300K$.
   ii) Assuming that $C_1\approx 2C_2$, estimate the position of the Fermi level E_F relative to the middle of the bandgap.

   For silicon : Atomic density : $5 \times 10^{22}\ cm^{-3}$
   Intrinsic concentration: $1.45 \times 10^{10}\ cm^{-3}$ at $T=300K$
   $E_g= 1.11eV$ at $T=300K$

QUESTION 8 **: SOLID STATE PHYSICS II**

Heating of a platinum wire

We are interested in using the electrical resistance of a platinum wire (Pt) as a temperature sensor at high temperature. Let the wire have a resistivity ρ_0 = 10 $\mu\Omega$–cm at a reference temperature of T_0 = 25 °C, and let the resistivity be linear with the temperature, following

$$\rho(T) = \rho_0\,[1 + \alpha(T - T_0)],$$

where α = 3,85 x 10^{-3} °C^{-1} between -200 et 700 °C. We wish to make sure that the electrical current do not produce any significant heating of the wire due to Joule effect. The platinum wire, with a diameter of 100 μm and a length of 10 cm is submitted to a voltage *V* of 1 mV for a duration of 10 seconds. Assuming that we can neglect the cooling of the wire (surface heat radiation), the equation describing the evolution of the temperature is

$$MC\frac{dT}{dt} = \frac{V^2}{R}.$$

where *M* is the wire mass, *C* its specific heat and *R* its electrical resistance. We further assume that the specific heat in the temperature range of interest (500 à 700 °C) is dominated by the contribution of the network (phonons). According to Debye's model, the internal energy of a solid of *N* atoms is

$$U = 9Nk_B\frac{T^4}{\theta_D^3}\int_0^{\theta_D/T}\frac{x^3}{e^x - 1}dx\,.$$

Platinum has a density of 21.45 g/cm^3, an atomic mass of 195,1 *u* and a Debye temperature of 240 K. From these data and equations:

a) **(5 points)** Determine an approximated expression for the temperature change as a fonction of *V*, *M*, *R*, C and of time, assuming that the parameters *M, C* and *R* are temperature independent.

b) **(5 points)** Determine an approximated expression for the specific heat (J/kg°C) at high temperature as compared to Debye's temperature ((Dulong & Petit law).

c) **(8 points)** Calculate the numerical value of the temperature elevation due to Joule effect at 600 °C.

d) **(2 points)** Briefly discuss, based on your result found in c), whether the temperature independent parameters approximation used in a) is reasonable.