EXAMEN GÉNÉRAL DE SYNTHÈSE – ÉPREUVE ÉCRITE
Programme de doctorat en génie physique

Jeudi 20 novembre 2014

Salle A-532

de 9h30 à 13h30

NOTES :

- *No documentation allowed.*
- *A non-programmable calculator is allowed.*
- *The candidate can answer up to 6 questions of his choice.*
- *Each question is worth 20 points.*
- ***Use a notebook different for each question, making sure to include the question number on it.***
- *This questionnaire contains 8 questions, 13 pages.*

ENGLISH VERSION

Département de génie physique
Pavillon principal
Téléphone : 514-340-4787
Télécopieur : 514-340-3218
Courriel : info@phys.polymtl.ca

Adresse postale
C.P. 6079, succ. Centre-ville
Montréal (Québec) Canada H3C 3A7
www.polymtl.ca

Campus de l'Université de Montréal
2900, boul. Édouard-Montpetit
2500, chemin de Polytechnique
Montréal (Québec) Canada H3T 1J4

CONSTANTES PHYSIQUES :

$e = 1.602 \; x \; 10^{-19}$ C

$m_e = 9{,}109 \; x \; 10^{-31}$ Kg

$\hbar = 1.055 \; x \; 10 \text{ x } 10^{-34}$ J·s

$k_B = 1.381 \; x \; 10^{-23}$ J/K

$\varepsilon_0 = 8.854 \text{ x} 10^{-12} \text{ F/M}$

$\mu_0 = 4\pi \text{ x } 10^{-7} \text{ N/A}^2$

ÉQUATIONS PHYSIQUES :

$$\nabla \cdot \mathbf{D} = \rho_f$$

$$\nabla \cdot \mathbf{B} = 0$$

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$$

$$\nabla \times \mathbf{H} = \mathbf{J}_f + \frac{\partial \mathbf{D}}{\partial t}$$

$$\mathbf{B} = \frac{\mu_0}{4\pi} \int \frac{\mathbf{J} \times \hat{\mathbf{r}}}{|\mathbf{r}|^2} dV$$

$$f_{FD}(E) = \frac{1}{e^{(E-\mu)/k_B T} + 1}$$

$$f_{BE}(E) = \frac{1}{e^{(E-\mu)/k_B T} - 1}$$

ÉQUATIONS MATHÉMATIQUES

Intégrale

$$\int_0^\infty \sqrt{x} \; e^{-x} \, dx = \frac{\sqrt{\pi}}{2}$$

$$\int_0^\infty e^{-ax^2} \, dx = \frac{1}{2}\sqrt{\frac{\pi}{a}}$$

Loi des cosinus

$$c^2 = a^2 + b^2 - 2ab \cos \gamma$$

Approximation de Stirling

$$n! \approx n^n e^{-n} \sqrt{2\pi n}$$

Identité

$$\nabla^2 \mathbf{A} = \nabla(\nabla \mathbf{A}) - \nabla \times \nabla \times \mathbf{A}$$

Identités trigonométriques

$$\sin(\alpha \pm \beta) = \sin \alpha \cos \beta \pm cos \; \alpha \sin \beta$$

$$\cos(\alpha \pm \beta) = \cos \alpha \; cos \; \beta \mp sin \; \alpha \; sin \; \beta$$

$$\sin 2\theta = 2 \sin \theta \cos \theta$$

$$\cos 2\theta = \cos^2 \theta - \sin^2 \theta$$

$$\cos \alpha \cos \beta = \frac{1}{2}\left(\cos(\alpha + \beta) + \cos(\alpha - \beta)\right)$$

$$\sin \alpha \sin \beta = \frac{1}{2}\left(\cos(\alpha - \beta) - \cos(\alpha + \beta)\right)$$

$$\sin \alpha \cos \beta = \frac{1}{2}\left(\sin(\alpha + \beta) + \sin(\alpha - \beta)\right)$$

QUESTION 1 : **ELECTRICITY AND MAGNETISM**

Magnetic quadrupole

Four electrical wires in a perfect square configuration conduct a DC current of the same magnitude. The current circulates in the same direction in one opposite pair of wires, and in the opposite direction in the other pair. In Figure 1 are shown the configuration of the wires and the sign of the current.

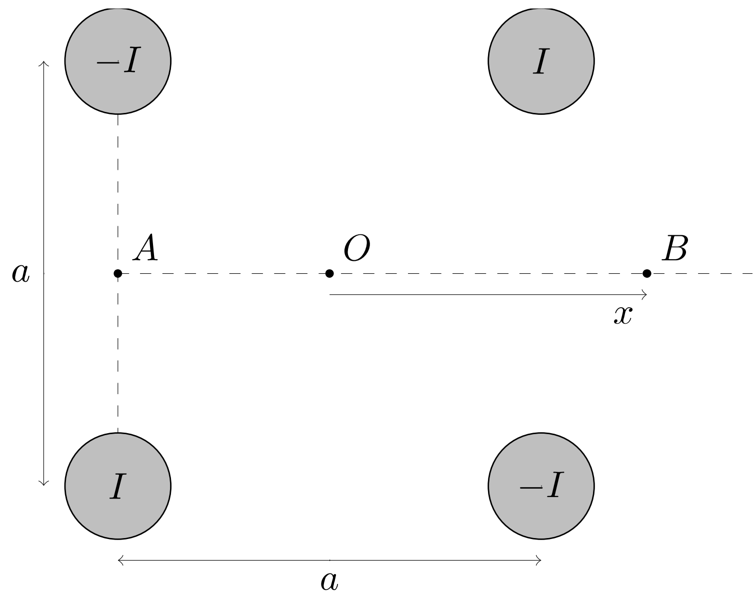


Figure 1 — Cross-section of four electrical wires in a square configuration, each side of dimension *a*, centered at the origin *O*. The current in each wire is indicated, a positive current circulating out of the plane towards the observer. Point *A* is situated mid-way between the two wires on the left, point *O* at the origin and at the center of the square, and point *B* at an arbitrary distance *x* of the origin, on a bisector of the square.

Answer the following questions.

1. **(5 Pts)** Calculate the magnetic field $\vec{H}$ at the origin *O*.

2. **(5 Pts)** Calculate the magnetic field $\vec{H}$ at point *A*.

3. **(5 Pts)** Calculate the magnetic field $\vec{H}(x)$ at point *B*, assumed to lie far from the origin. We are therefore looking for an asymptotic expression for $x \gg a$.

4. **(5 Pts)** Compare this asymptotic expression with one that would be obtained for a magnetic dipole. By “dipole”, we mean two wires separated by a distance *a* and carrying opposite currents of magnitude 2*I*. Cite a potential advantage of using the quadrupole configuration for electrical energy transport.

***QUESTION* 2** : **QUANTUM MECHANICS**

Quantum cascade lasers take advantage of electron energy levels of a complex semiconductor structure composed of superlattices (electron injectors) and quantum wells (emitters). The operating principle is the following :

- an electric field moves electrons through the structure,
- electrons from the injector (the « injecteur » is shown in gray; its nature is not important here) tunnel to the level *n = 3* of the quantum well (shown in black),
- electrons relax to the fundamental level by light emission. With proper feedback, stimulated emission occurs between levels *n = 3* and *n = 2*,
- electrons are then extracted from the quantum well and transferred to the next injector,
- stimulated emission occurs again in a second quantum well and this cycle is repeated 5 to 50 times.

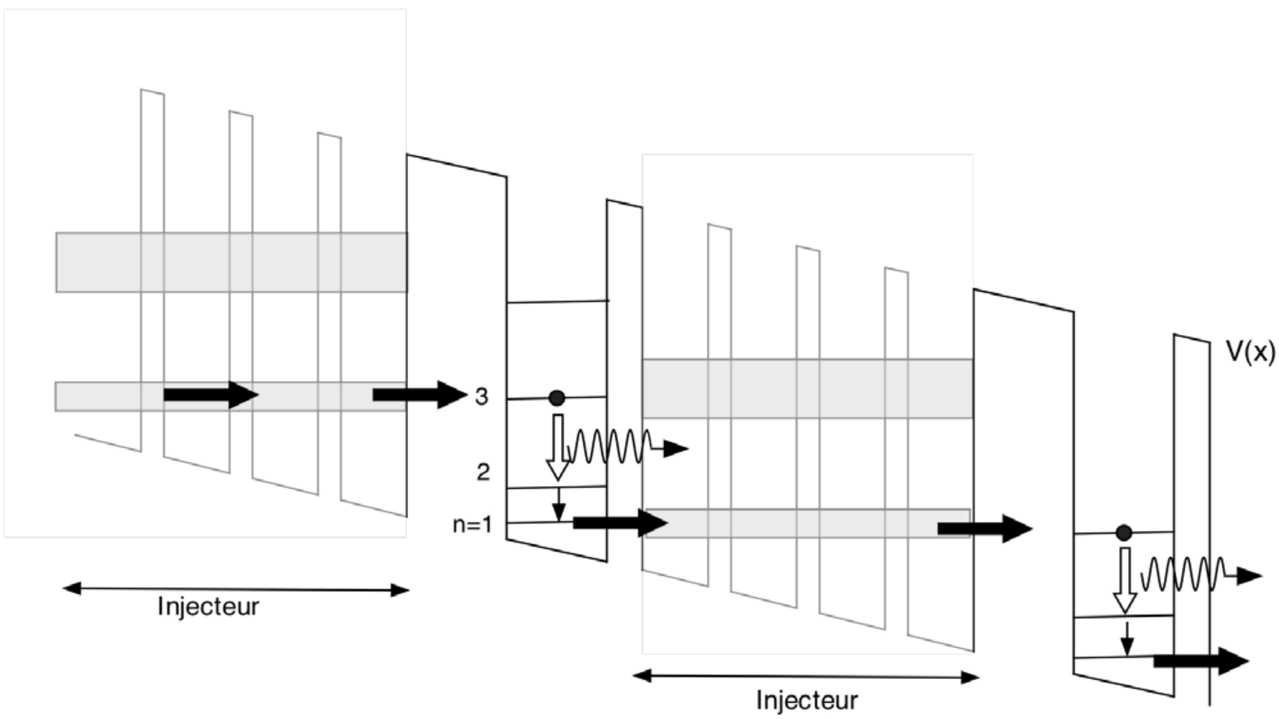


Fig. 1 – Quantum cascade laser. The rectangular profile shows the potential *V(x)* produced by the semiconductor materials under the effect of an external electric field. While transiting through the structure, the electron relaxes by spontaneous and stimulated emission. The injector (« injecteur ») plays an important role in the operation of this laser, but it is not important for this problem.

(see next page)

***QUESTION* 2** : **QUANTUM MECHANICS**

a) **(5 points)** For a well width of 8 nm and an effective mass of $m_e = 6.38 \times 10^{-32}$ kg for this material, provide a Hamiltonian that you can solve and estimate the laser operation frequency.

b) **(5 points)** What are the approximations you have used to get this estimate? Discuss their relative importance.

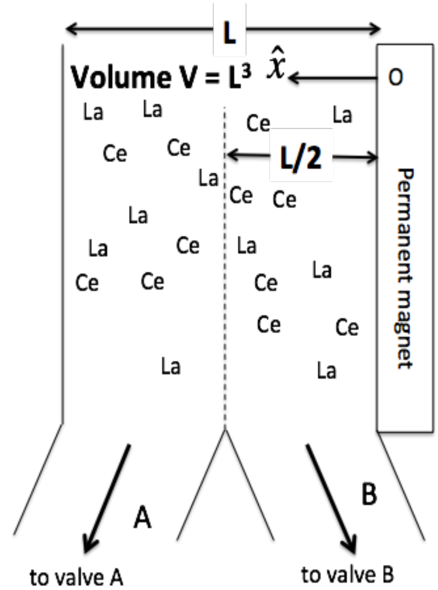
c) **(5 points)** How good is this estimate? Does it overestimate or underestimate the frequency?

d) **(5 points)** If you had a computer and a few hours to solve this problem, how would you proceed to find a more accurate solution?

QUESTION 3 : STATISTICAL PHYSICS

Magnetic separation of rare earth particles

In a mining process, we seek to magnetically separate two types of rare earth, lanthanum (diamagnetic) and cerium (paramagnetic), present as nanoparticles suspended in a fluid. The separator used is made of a cubic chamber of volume $V = L^3$ on the side of which a permanent magnet of surface L^2 can be placed (see Fig.). At the beginning of the separation process, the magnet is in place. We admit that it will generate a constant magnetic force on the particles everywhere in the fluid. Rare earth particles will migrate to reach an equilibrium position forming a trade off between diffusion (thermal agitation), which will tend to disperse particles and the magnetic force, which will tend to concentrate them. Once equilibrium is achieved, we open the valves located below the separator and the fluid is purged in two equal volumes through the exits A and B. The solutions from these exits are analyzed and the enrichment of rare earths assessed.



Problem-specific considerations:

- The equilibrium of a solute submitted to thermal agitation and external forces can be treated as that of an ideal gas in the presence of external forces.

- The system's temperature is constant throughout the process.

- Gravity can be neglected as well as particle-particle interactions.

- The relative susceptibilities of *La* and *Ce* are identical but of opposed signs ($\chi_{Ce} = -\chi_{La} = \chi$)

- The expression of the magnetic force generated by the magnet is given by $\hat{F}_{m,i} = -\chi_i\, f\, \hat{x}$ where χ_i is the susceptibility of element *i* and *f* is a positive constant with units of force.

- The enrichment coefficient of cerium with respect to lanthanum, after a separation process, is defined as

$$E_{Ce} = \frac{P_{Ce}\,(0 < x < L/2)}{P_{La}\,(0 < x < L/2)}$$

  where P_{Ce} and P_{La}, are the probabilities of finding the rare earths in the separator's half closest to the magnet.

(see next page)

QUESTION 3 : **STATISTICAL PHYSICS**

Questions:

A) **(10 Pts)** For each component, give the system's partition function before and after the application of a magnetic field.

B) **(5 Pts)** Give the expression of the probability of finding a cerium particle at a distance *x* from the magnet. Give also this expression for lanthanum.

C) **(5 Pts)** Give, as a function of k_B, T, χ, f, *et* L, the expression of the enrichment coefficient E_{Ce} in the fluid volume recovered through valve B.

Potentially useful integral:

$$\int_{-\infty}^{\infty} e^{-t^2} dt = \sqrt{\pi}$$

QUESTION 4 : **CLASSICAL MECHANICS**

An Astronaut lands on an ellipsoidal shaped planet called α-Amusant in the Cerebral galaxy. The planet has a semi-major axis, *b*, and semi-minor axis, *a*, in the ratio of 2:1. The semi-major axis, *b*, has a length of 6371km. The mass of the planet m_α is 5.97 x 10^{24} kg. This planet is spinning at 1 rotation per hour in an anti-clockwise direction as shown by the curved arrow in Fig. 1, in the plane of the paper around the axis at point 'O'.

The Astronaut, Captain Sandra, whose mass is *m*, lands close to the point *P* shown in Figure 1, and begins to walk on the surface (in the plane of the paper) in a clockwise direction (exactly opposite direction to the rotation of the planet). Her walking speed is vey slow compared to the velocity associated to the rotation.

After several months of walking, Captain Sandra arrives at a point at which she just lifts off the surface. The universal Gravitational constant, *G* is $6.7 \times 10^{-11} m^3 \, kg^{-1} \, s^{-2}$.

Hints: To simplify calculations, assume that the mass of α-Amusant is all concentrated at point O.

1. **(8 Pts)** What is the distance *r* from the center of α-Amusant at which she becomes weightless?

2. **(4 Pts)** What is her velocity at that point?

3. **(4 Pts)** What will happen to Cpt. Sandra if she does not immediately run back where she came from. Explain your answer.

4. **(4 Pts)** Can there be an atmosphere on α-Amusant? Justify your answer.

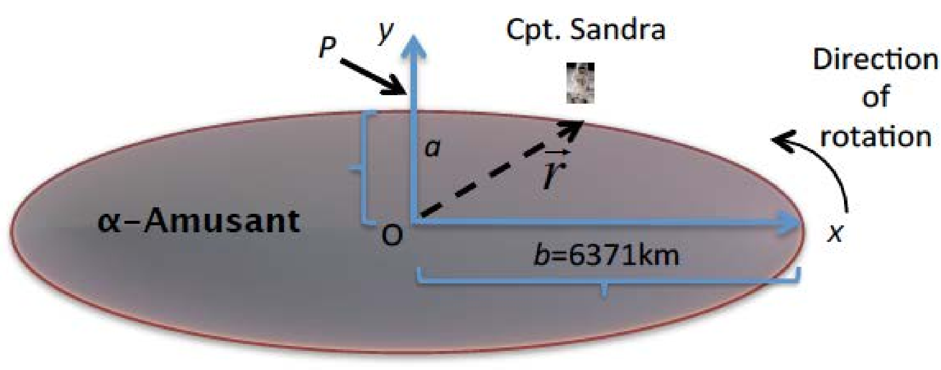


Fig. 1 Planet α-Amusant with astronaut, Cpt Sandra

QUESTION 5 : **OPTICS I**

1.1 (2,5 points) Fermat Principle

Describe in your own words Fermat's Principle.

1.2 (7,5 points) Refraction

Use Fermat's Principle to obtain Snell's law for refraction using the variables shown in Figure 1.

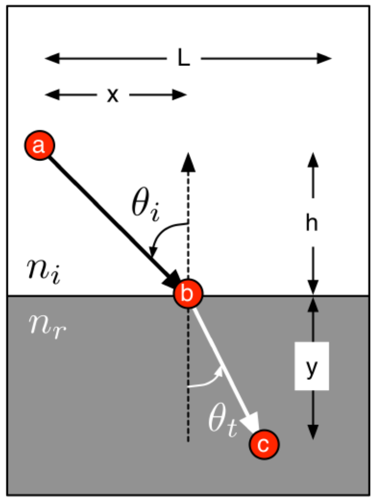


Figure 1 Refraction

Note: Assume that points a, b and c all lie in a plane normal to the interface

1.3 (10 points) Gradient index lens - GRIN

Contrarily to conventional lenses using a constant index of refraction and curved faces, gradient index lenses use planar faces ($R \to \infty$) and index of refraction which varies radially according to $n(r) = n_{max}\left[1 - \frac{Ar^2}{2}\right]$, where A is a positive constant and n_{max} is the refractive index at the position $r = 0$. Using Fermat's Principle, show that this index profile results in a positive lens of thickness d and focal length f. In addition, obtain an expression for A as a function of d and of f. For this demonstration, use the binomial approximation $\sqrt{1+x^2} \cong 1 + \frac{x^2}{2}$ as well as the paraxial approximation which states that for small angles, and small thickness, the optical path of a ray within the lens is given by $n(r) \cdot d$.

QUESTION 6 : **OPTICS 2**

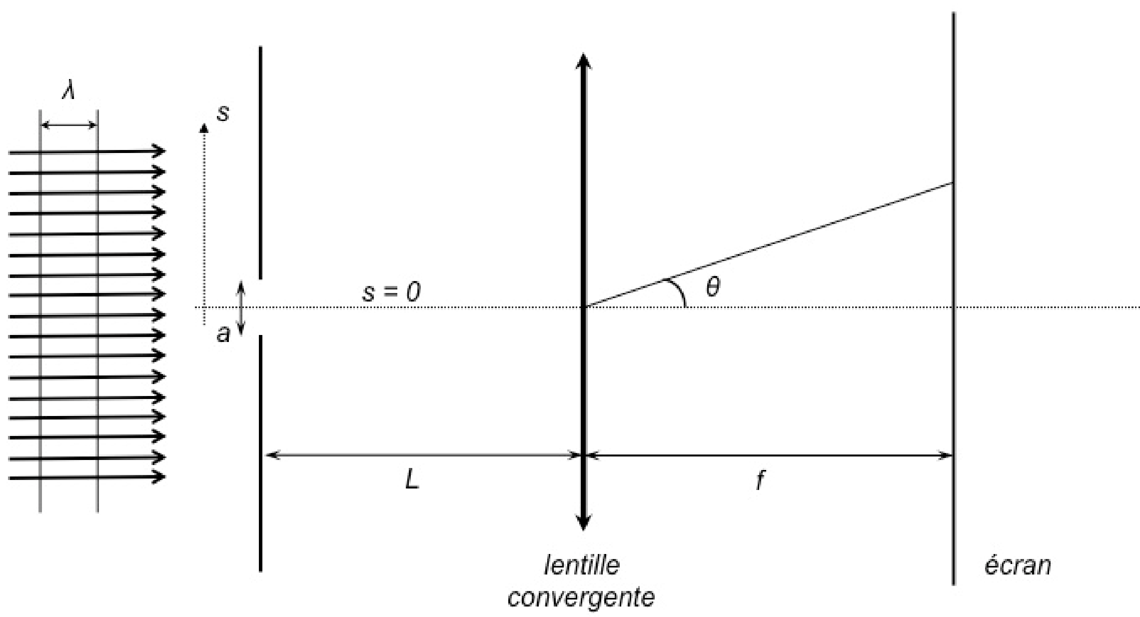
LIGHT INTERFERENCE

(a) **(12 Pts)** A light beam composed of plane waves with wavelength λ is incident on a rectangular opening (a slit) of height *a* and width *b*. The width *b* is not shown in the figure because it is in the direction perpendicular to the page. A converging lens of focal length *f* is placed at a distance L from the aperture (the slit) and a viewing screen is placed at a distance *f* (focal length) to the right of the lens.

Calculate the mathematical expression representing the light intensity *I* (diffraction pattern) that will be observed on the screen.

Notes :

- You should assume that the distance $L + f$ is large enough so that the light striking the screen can be approximated as being composed of planar wavefronts.
- You must calculate the diffraction pattern using the method based on the integral of contributions spherical wavelets emitted from the opening.
- The pattern on the screen is in two dimensions but only the direction *s* is shown in figure: therefore, only one angle (ϑ) is shown which allows to vary the observation point on the screen in the direction *s*. Although only one angle is shown in the figure, to calculate the pattern on the screen you must enter a second angle (φ) to vary the observation point in the direction transverse to *s*.

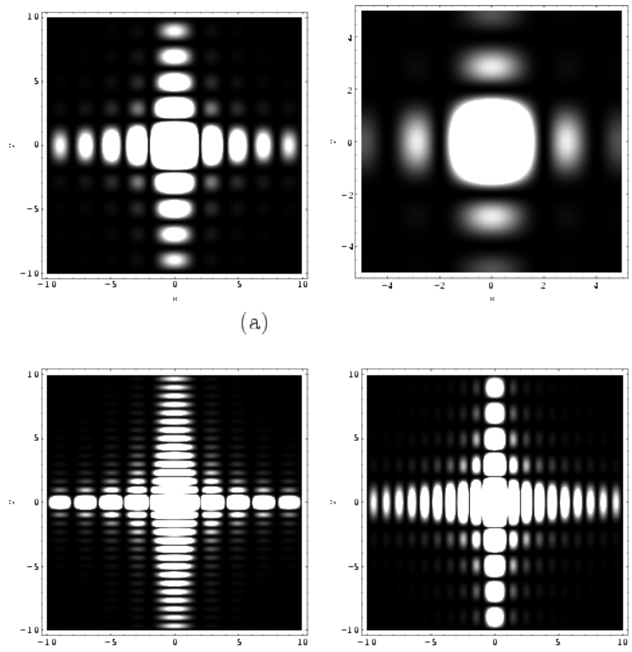


Translation of words in the figure: 'lentille convergente' is 'converging lens', 'écran' is 'screen'.

(see next page)

QUESTION 6 : **OPTICS 2**

(b) **(8 Pts)** What are the physical parameters explaining the different appearance of the 4 following diffraction patterns. Explain qualitatively how these parameters vary from one image to another.



(a) (b) (c) (d)

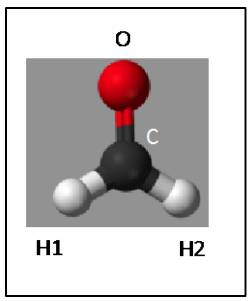
QUESTION 7 :   **SOLID STATE PHYSICS I**

The polarizability (α) of a system results from several contributions. Among them are the electronic, ionic, and dipolar contributions.

1) **(5 Pts)** Give the definition of polarizability and explain the meaning of electronic, ionic, and dipolar polarizability.

2) **(4 Pts)** State whether each of the following is an atomic property or a macroscopic property: dipolar moment, polarizability, and dielectric constant. Give the SI units of each of these properties.

3) **(3 Pts)** Calculate the dipole moment of the molecule in the figure below considering that the partial charge on each of the atoms H is +0.19 (in *e* units), the partial charge on the atom C is 0, and the partial charge on the atom O is -0.38 and that the positions of the atoms are illustrated in the figure (H1 (-94 pm, -61 pm, 0 pm), H2 (94 pm, -61 pm, 0 pm), C (0 pm, 0 pm, 0 pm), O (0 pm,118 pm, 0 pm).



4) **(6 Pts)** Suppose that the same molecule as in 3) approaches an anion. What is the favorable orientation of the molecule? Calculate the electric field (in volts per meter) experienced by the anion when the dipole is (a) 1.0 nm, (b) 10 nm, (c) 30 nm from the ion.

5) **(2 Pts)** The polarizability volume ($\alpha' = \alpha / 4\pi\varepsilon_0$ *with* $\varepsilon_0 = 8.85419\ 10^{-12}\ C^2 \cdot m^{-1} \cdot J^{-1}$) of H_2O at optical frequencies is $1.5 \cdot 10^{-24}$ cm^3: estimate the refractive index of water. The experimental value is 1.33.

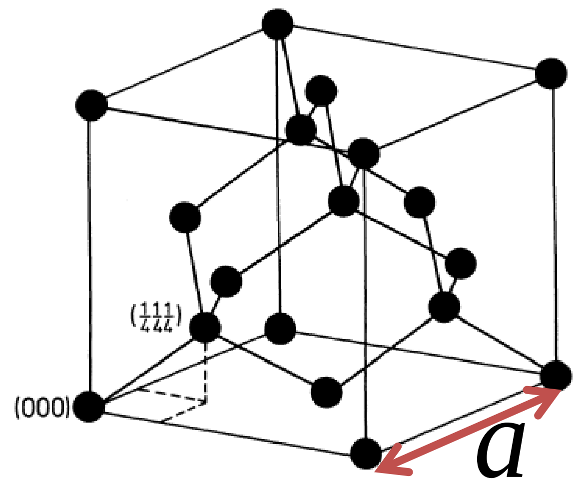
Clausius-Mossotti relation

$$\frac{\varepsilon - 1}{\varepsilon + 2} = \frac{1}{3\varepsilon_0} N\alpha$$

1 D = 3.335 10^{-30} C·m
ε_0 = 8.85 10^{-12} F/m

QUESTION 8 : **SOLID STATE PHYSICS 2**

A homogeneous elementary semiconductor has the diamond crystallographic structure (a face centered cubic lattice of parameter $a = 0.5\ nm$ with a base composed of two atoms, which supports tetrahedral bonds, see figure). Crystal purification can be achieved to the level of 10 ppm (part per million) of residual impurities. The residual impurities are donors with ionization energy of *12.5 meV)*. At *300 K* the energy gap of the semiconductor is $E_g = 1.2$ *eV* , the carrier mobility is $0.1\ m^2/Vs$, and the intrinsic carrier concentration is less than $10^{16}\ m^{-3}$.



(a) **(10 Pts)** Write the charge neutrality equation assuming that all the impurities are ionized, and determine the majority carrier concentration at *300 K.* Justify the assumption as well as all the approximations which are necessary to evaluate the carrier concentration.

(b) **(6 Pts)** Give the expression for the electrical resistivity and estimate its value in the dark (no illumination). Compare with the order of magnitude of the intrinsic resistivity.

(c) **(4 Pts)** A monochromatic laser beam (of wavelength $\lambda = 1.55\ \mu m$) illuminates this semiconductor with an intensity of 1 mW/cm^2 . Indicate if the resistivity will (i) drastically decrease, (ii) slightly increase, or (iii) slightly decrease. Justify.

Note : Clearly indicate your assumptions and show your work.