POLYTECHNIQUE MONTRÉAL
LE GÉNIE EN PREMIÈRE CLASSE

EXAMEN GÉNÉRAL DE SYNTHÈSE – ÉPREUVE ÉCRITE
Programme de doctorat en génie physique

<u>Jeudi 23 novembre 2017</u>

Salle B-406

de 9h30 à 13h30

<u>NOTES</u> :

- *No documentation allowed.*
- *A non-programmable calculator is allowed.*
- *The candidate answers up to 6 questions of his choice among 8.*
- *Each question is worth 20 points.*
- ***Use a different notebook for each question, making sure to include the question number on it.***
- *This questionnaire contains 9 questions, 10 pages.*

ENGLISH VERSION

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$$\sum_{n=0}^{\infty} x^n = \frac{1}{1-x}, \quad |x|<1$$

Formulas and conversion

$$\int_{-\infty}^{\infty} \Psi_v^* \Psi_v dx = \alpha \int_{-\infty}^{\infty} \Psi_v^* \Psi_v dy = \alpha \int_{-\infty}^{\infty} H_v^2(y) e^{-y^2} dy = \alpha \pi^{1/2} 2^v v!$$

v !=v(v-1)(v-2)…

1 u = 931,494 MeV/c^2

CONSTANTES PHYSIQUES

$e = 1.602 \times 10^{-19}$ C
$m_e = 9{,}109 \times 10^{-31}$ Kg
$\hbar = 1.055 \times 10 \times 10^{-34}$ J·s
$k_B = 1.381 \times 10^{-23}$ J/K
$\varepsilon_0 = 8.854 \times 10^{-12}$ F/M
$\mu_0 = 4\pi \times 10^{-7}$ N/A^2

ÉQUATIONS PHYSIQUES :

$$\nabla \cdot \mathbf{D} = \rho_f$$

$$\nabla \cdot \mathbf{B} = 0$$

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$$

$$\nabla \times \mathbf{H} = \mathbf{J}_f + \frac{\partial \mathbf{D}}{\partial t}$$

$$\mathbf{B} = \frac{\mu_0}{4\pi} \int \frac{\mathbf{J} \times \hat{\mathbf{r}}}{|\mathbf{r}|^2} dV$$

$$f_{FD}(E) = \frac{1}{e^{(E-\mu)/k_B T} + 1}$$

$$f_{BE}(E) = \frac{1}{e^{(E-\mu)/k_B T} - 1}$$

Relation de Clausius-Mossotti

$$\frac{n\alpha}{3\varepsilon_0} = \frac{\varepsilon_r - 1}{\varepsilon_r + 2}$$

EQUATIONS MATHÉMATIQUES

Intégrale

$$\int_0^{\infty} \sqrt{x}\, e^{-x}\, dx = \frac{\sqrt{\pi}}{2}$$

$$\int_0^{\infty} e^{-ax^2}\, dx = \frac{1}{2} \sqrt{\frac{\pi}{a}}$$

$$\int_0^{\infty} x^2\, e^{-ax^2} dx = \frac{\sqrt{\pi}}{4a^{3/2}}$$

Loi des cosinus

$$c^2 = a^2 + b^2 - 2ab \cos \gamma$$

Approximation de Stirling

$$n! \approx n^n e^{-n} \sqrt{2\pi n}$$

Identité

$$\nabla^2 \mathbf{A} = \nabla(\nabla \mathbf{A}) - \nabla \times \nabla \times \mathbf{A}$$

Identités trigonométriques

$$\sin(\alpha \pm \beta) = \sin\alpha \cos\beta \pm \cos\alpha \sin\beta$$
$$\cos(\alpha \pm \beta) = \cos\alpha \cos\beta \mp \sin\alpha \sin\beta$$

$$\sin 2\theta = 2 \sin\theta \cos\theta$$
$$\cos 2\theta = \cos^2\theta - \sin^2\theta$$

$$\cos\alpha \cos\beta = \frac{1}{2}\left(\cos(\alpha+\beta) + \cos(\alpha-\beta)\right)$$

$$\sin\alpha \sin\beta = \frac{1}{2}\left(\cos(\alpha-\beta) - \cos(\alpha+\beta)\right)$$

$$\sin\alpha \cos\beta = \frac{1}{2}\left(\sin(\alpha+\beta) + \sin(\alpha-\beta)\right)$$

QUESTION 1 : ELECTRICITY AND MAGNETISM

Electricity and magnetism: Penning trap

The Penning trap constitutes one way of confining charged particles in a vacuum chamber. It is used, for example, to store anti-particles such as anti-protons. The trap comprises an electric quadrupole for the axial confinement. We are interested here in the properties of this quadrupole in the vicinity of its center.

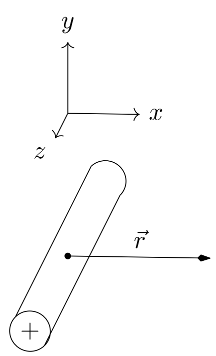


Figure 1: Uniformly charged infinite wire.

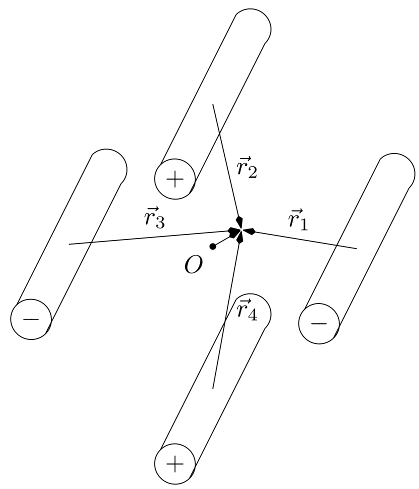


Figure 2: Quadrupole comprising four wires crossing the plane $z = 0$. All vectors shown, as well as the origin, lie in that plane.

1) **(5 pts)** Write an expression for the electric field at a distance $|\vec{r}|$ of an infinitely long single wire uniformly charged with charge density λ, as shown in Figure 1.
2) **(5 pts)** Write an expression for the electric field at any point on the xOy plane at $z = 0$, as a function of its relative position to the four wires $\vec{r}_1$, $\vec{r}_2$, $\vec{r}_3$ and $\vec{r}_4$, as shown in Figure 2.
3) **(5 pts)** Plot a diagram representing qualitatively the electric field lines in the xOy plane in the vicinity of the center of the quadrupole.
4) **(5 pts)** We define the cartesian coordinate system $\vec{r} = x\hat{x} + y\hat{y}$ having its origin at the center of the quadrupole at a distance R of each wire. Find an expression for the electric field $\vec{E}(x, y)$ valid near the origin as a first order series expansion in x and y.

The following series expansion is provided.

$$\frac{1}{1+x} \approx 1 - x + x^2 - \cdots$$

$$|\hat{x}| = 1 \quad |\hat{y}| = 1$$

QUESTION 2 : QUANTUM MECHANICS

The 2*s* (n = 2, l = 0, m_l = 0) and the 2*p* (n = 2, l = 1, m_l = 0) states of the electron in the hydrogen atom are described, respectively, by the following wavefunctions:

state 2*s*:

$$\psi_{2s}(r,\theta,\varphi) = A_o\left(2-\frac{r}{a_o}\right)e^{-\left(\frac{r}{2a_o}\right)}$$

state 2*p*:

$$\psi_{2p}(r,\theta,\varphi) = \left[\frac{1}{2\sqrt{6}\,a_o^{3/2}}\left(\frac{r}{a_o}\right)e^{-\left(\frac{r}{2a_o}\right)}\right]\left[\frac{\sqrt{6}}{2}\cos(\theta)\right]\left[\frac{1}{\sqrt{2\pi}}\right]$$

where a_o is the Bohr's radius. Based on these wavefunctions, you must:

a) **(5 pts)** Briefly explain the methodology that we could use to solve the Schrödinger equation and thus, obtain the wavefunctions Ψ_{2s} and Ψ_{2p};

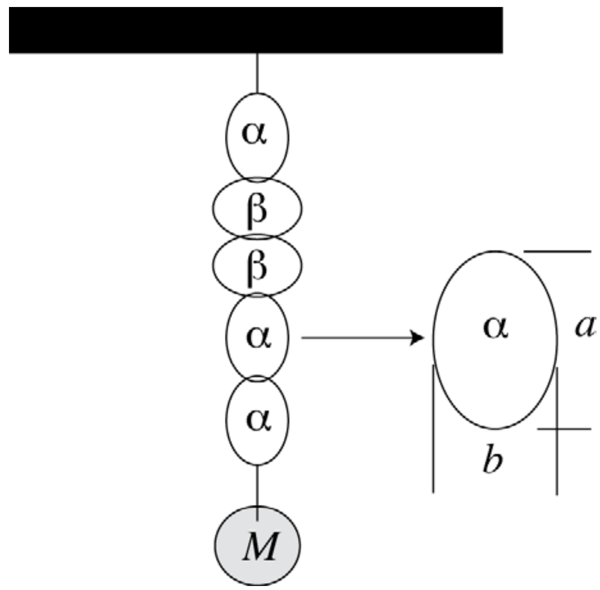
b) **(5 pts)** Calculate the value of A_o for the wavefunction corresponding to the 2*s* state and explain the approach adopted to carry out your calculations;

c) **(5 pts)** Estimate the most probable position of the electron at the 2*p* state, with respect to the nucleus;

d) **(5 pts)** Identify and explain the degeneracy in the 2*s* and 2*p* states, when the electron is described by the wavefunctions Ψ_{2s} and Ψ_{2p}.

QUESTION 3 : **STATISTICAL PHYSICS**

A vertical chain, that supports a mass M, is composed of N massless links (we will assume that N is very large). This chain is tied to a support at the top (see figure below).



Each link, which has the form of an ellipsis, can take two different configurations in the chain, noted α and β, that have respective lengths a et b. These correspond to the major and minor axis of the ellipsis. The energy of the system is given by $E = MgL$ where Mg is the gravitational force exerted by the mass and L the length of the chain.

a) **(8 pts)** Give an expression for the entropy $S(E, N)$ of this chain.

b) **(8 pts)** What is the statistical temperature associated with this chain.

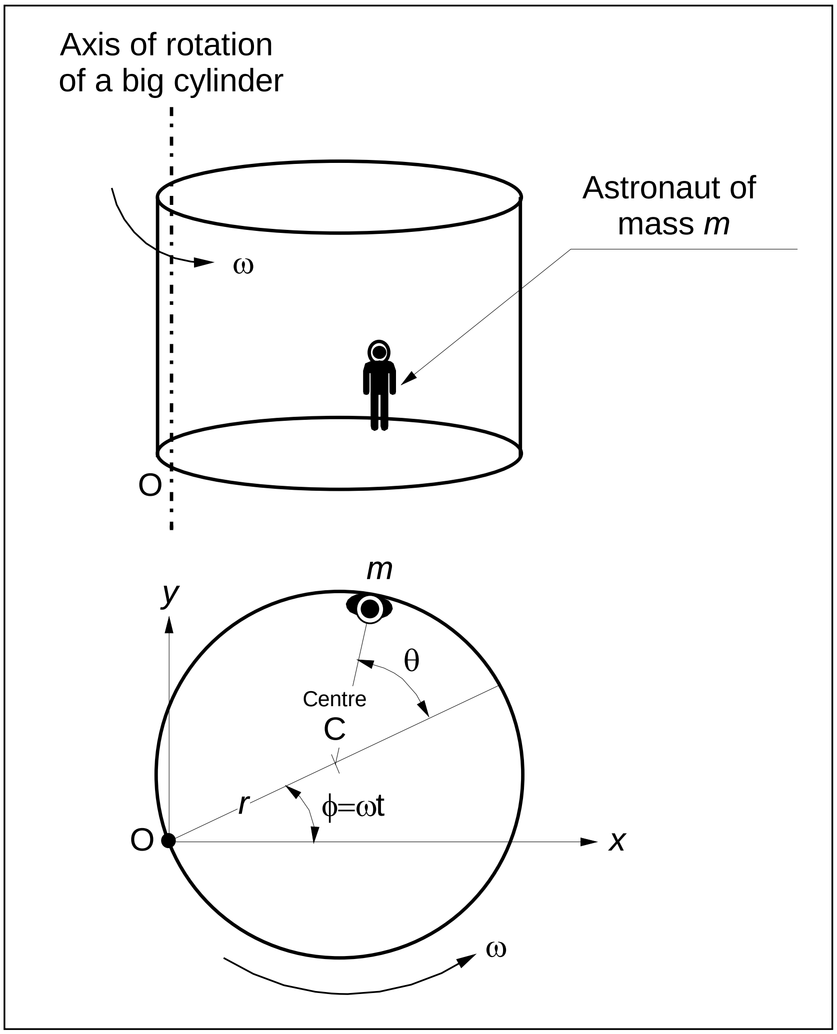
c) **(4 pts)** Find an expression for the energy chain as well as the number of links in configuration α at very high temperature ($T \to \infty$).

QUESTION 4 : CLASSICAL MECHANICS

In order to train future astronauts for a very long journey to Mars, a physical engineer proposes the use of a very big cylinder rotating eccentrically with a constant angular velocity ω with respect to an axial axis, as shown in the figure. If an astronaut placed vertically inside the cylinder can be considered very small with respect to the dimensions of the system (i.e. he can be considered as a point particle of mass m), and he can move freely without friction up to the inner wall, you must:

a) **(12 pts)** Determine the equation of motion of the astronaut as a function of θ (see the figure) by using Lagrange's formalism;

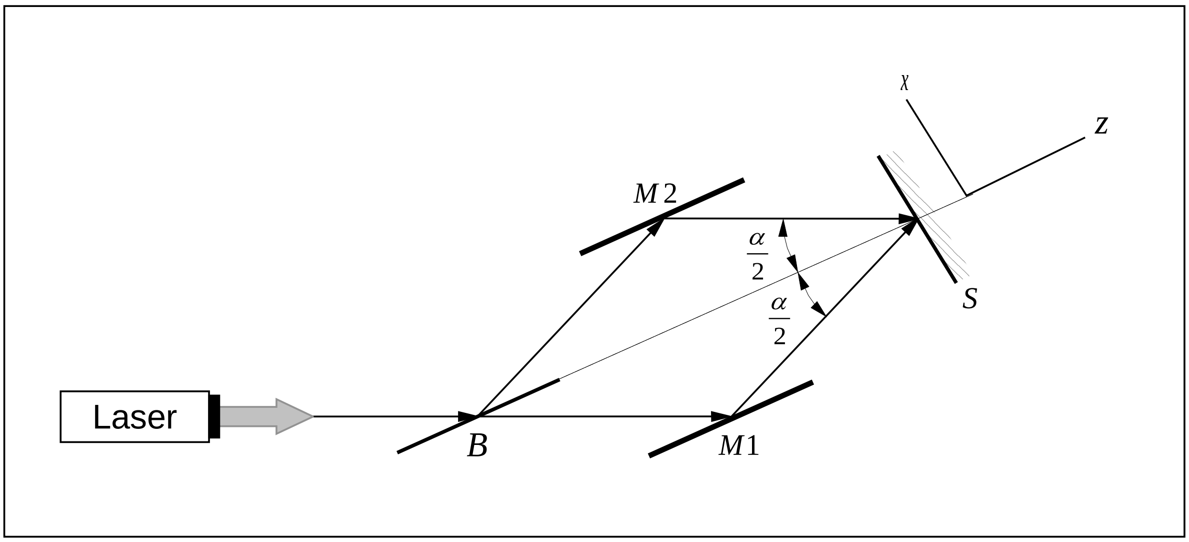
b) **(8 pts)** Demonstrate that the astronaut will follow a harmonic motion like a pendulum having a length $r = g/\omega^2$ (g = acceleration of gravity).



__QUESTION 5__ : **OPTICS I**

A wide non-diverging light beam from a laser source operating a wavelength, λ, is split into two using a 50% reflection beam splitter (B) and then brought together with two mirrors M1 & M2, as shown in Fig. 1. A vertical screen, S, is placed in the plane of intersection such that the normal to the screen bisects the angle of convergence, α, of the two rays.

1. **15 pts)** Find an expression for the period of the interference fringes (i.e. the distance between the bright regions of the interference pattern) seen on the screen,
2. **(5 pts)** If the wavelength of the laser is 500 nm and the angle, α = 36 degrees, calculate the period of the interference pattern.



QUESTION 6 : **OPTICS 2**

(a) (**10 pts)** A Young interference experiment is prepared as shown in the figure below. The slots are illuminated with a coherent light beam collimated with wavelength λ = 600 nm. When a thin plate made of a transparent dielectric material is placed behind one of the slits (at a very small distance $<\lambda/2$), the zeroth order fringe moves to the position previously occupied by the fourth order fringe.

You must find the thickness of the thin plate whose relative permittivity is $\varepsilon_r = 5$.

(b) (**10 pts)** You must now consider the same system, but this time with the thin plate removed. You must draw the interference pattern on the screen for a system consisting of two identical slits having a width **a** and separated by a center-to-center distance **g** when **g** / **a** = 5.

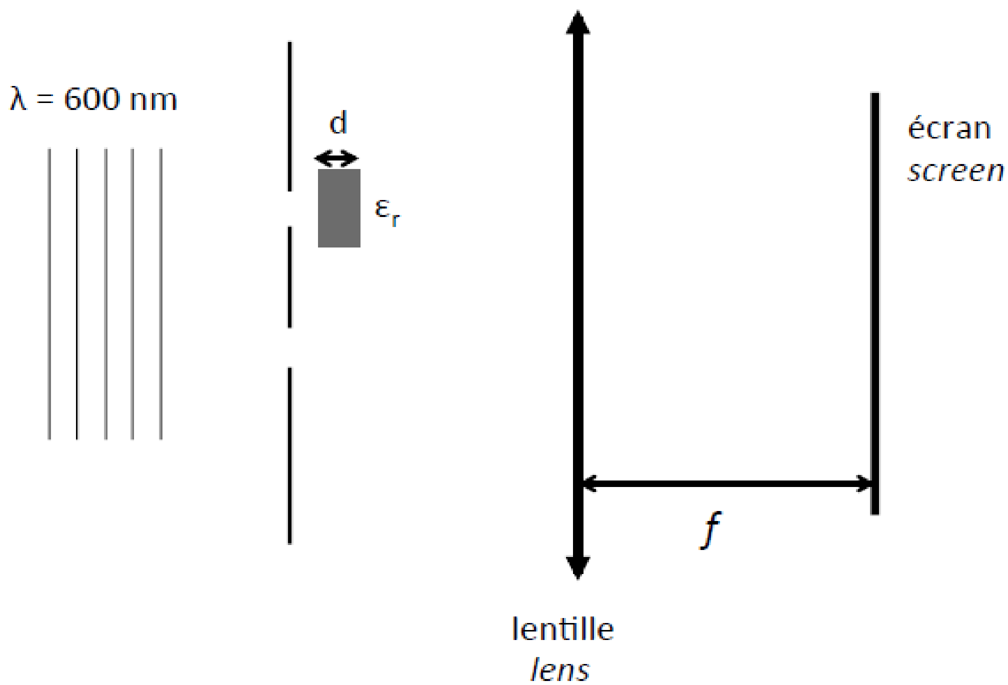


Figure caption:

System used to observe an interference pattern on the screen ('écran') through a lens ('lentille')

QUESTION 7 : **SOLID STATE PHYSICS I**

Silicon is an indirect bandgap semiconductor with a diamond-like lattice structure.

I. (5 pts) Structure :

I.1. Draw the silicon unit cell and indicate the inversion symmetry points.

I.2. Find silicon atomic density knowing that its lattice parameter is 0.543 nm.

I.3. Detailed investigations of point defects in silicon found that the vacancy formation energy is 1.0 eV. Find at which temperature the vacancy concentration reaches 5×10^{12} cm^{-3}.

II. (5 pts) Optical properties:

Silicon has acoustic and optical modes in its phonon spectrum. However, its optical modes are not IR-active in the sense that IR radiation does not excite phonons. Explain why.

III. (10 pts) Electronic Properties:

For an ideal silicon p-n junction:

III.1. Find the built-in potential at room temperature (RT) if the bulk resistivity in 1 Ω cm. Electron mobility in Si at RT is μ_n = 1400 $cm^2\ V^{-1}\ s^{-1}$; hole mobility is given by $\mu_p = \mu_n /3.1$; $n_i = 1.05 \times 10^{10}$ cm^{-3}.

III.2. Calculate the width of the space charge region for the applied voltages:

V = -10 V; 0 V; 0.3 V.

III.3. For the widths calculated in III.2 find the values of the corresponding maximum electric field. Compare these values with the electric field within a shallow donor: $E = \frac{e}{\varepsilon \times a_B^2}$, where Bohr's radius: $a_B = \frac{\varepsilon \hbar^2}{m_n^2 e^2}$ and $\frac{m_n}{m_0} = 0.33$.

Constants

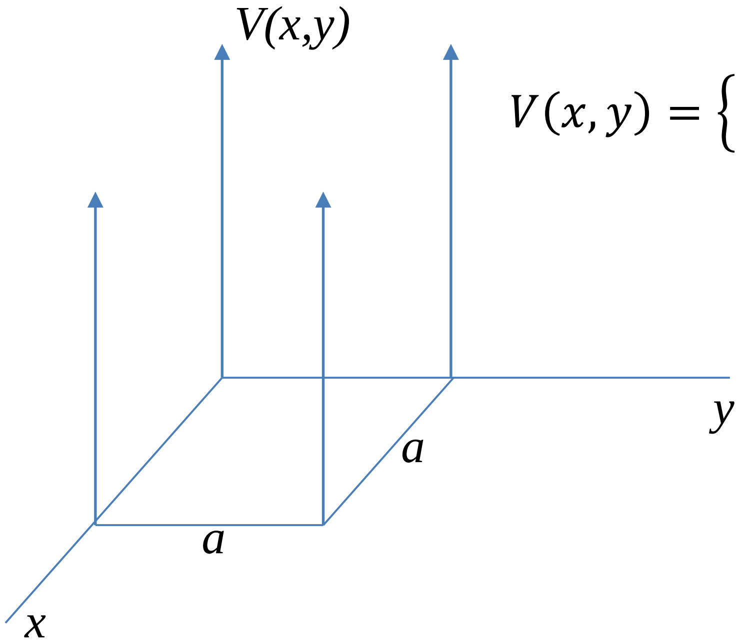
h = 6.62 10^{-34} m^2 kg s^{-1}; m_0 = 9.11 10^{-31} kg

Vacuum permittivity = 8.85 10^{-12} F/m

Si dielectric constant = 11.9.

QUESTION 8 : **SOLID STATE PHYSICS 2**

Question: Free electron gas



$$V(x,y) = \begin{cases} 0 & 0 < x < a,\ 0 < y < a \\ \infty & x \le 0, x \ge a, y \le 0, y \ge a \end{cases}$$

Consider a single electron in an infinite two-dimensional potential well of width *a*.

a) **(6 pts)** Derive the eigenstates. Show a diagram of the allowed quantum states in one quadrant of $\vec{k}$ space. Indicate the ground state of the electron on the diagram. What is the energy of this state?

Now consider *N* electrons in the same box.

b) **(3 pts)** Describe what principle one needs to use and what simplification to take in order to analytically calculate the resulting electronic distribution? Write down (no derivation required) the expression for the distribution and identify each term.

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c) **(4 pts)** Using a sketch similar to the one obtained in question (a), show which states are occupied if the total number of electrons in the box: $N = 16$. Assume $T = 0$ K. What is the energy difference between the occupied states of highest and lowest energy? Identify this energy difference for very large *N*.

d) **(5 pts)** Derive the density of states *D*(E) for very large *N*.

e) **(2 pts)** For very large *N*, sketch the density of occupied states for $T = 0$ K and a temperature $T > 0$. Label relevant energy scales.