POLYTECHNIQUE MONTRÉAL
TECHNOLOGICAL UNIVERSITY

QUALIFICATION EXAM – WRITTEN PART
Ph. D. Program in Engineering Physics

Thurdsay Novembre 22, 2018

Room B-530.3

from 9h30 to 13h30

NOTES :

- *No documentation allowed.*
- *A non-programmable calculator is allowed.*
- *The candidate answers to 6 questions of his choice among 8.*
- *Each question is worth 20 points.*
- ***Use a different notebook for each question, making sure to include the question number on it.***
- *This questionnaire contains 8 questions, 10 pages.*

ENGLISH VERSION

Department Engineering Physics
Main Building
Téléphone : 514-340-4787
Télécopieur : 514-340-3218
Courriel : info@phys.polymtl.ca

Adresse postale
P.O. Box 6079, Station Centre-ville
Montréal (Québec) Canada H3C 3A7
www.polymtl.ca

2900, boul. Édouard-Montpetit
Campus of the'Université of Montréal
2500, chemin de Polytechnique
Montréal (Québec) Canada H3T 1J4

$$\sum_{n=0}^{\infty} x^n = \frac{1}{1-x}, \quad |x|<1$$

PHYSICAL CONSTANTS

e = 1.602 x 10^{-19} C
m_e = *9,109 x 10*$^{-31}$ Kg
ħ = 1.055 x 10 x 10^{-34}J·s
k_B = *1.381 x 10*$^{-23}$ J/K
ε_0 = *8.854 x 10*$^{-12}$ F/M
$\mu_0 = 4\pi\ x\ 10^{-7}$ N/A^2

PHYSICS EQUATIONS :

$$\nabla \cdot \mathbf{D} = \rho_f$$

$$\nabla \cdot \mathbf{B} = 0$$

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$$

$$\nabla \times \mathbf{H} = \mathbf{J}_f + \frac{\partial \mathbf{D}}{\partial t}$$

$$\mathbf{B} = \frac{\mu_0}{4\pi} \int \frac{\mathbf{J} \times \hat{\mathbf{r}}}{|\mathbf{r}|^2} dV$$

$$f_{FD}(E) = \frac{1}{e^{(E-\mu)/k_B T} + 1}$$

$$f_{BE}(E) = \frac{1}{e^{(E-\mu)/k_B T} - 1}$$

Clausius-Mossotti relation

$$\frac{n\alpha}{3\varepsilon_0} = \frac{\varepsilon_r - 1}{\varepsilon_r + 2}$$

FORMULAS AND CONVERSION

$$\int_{-\infty}^{\infty} \Psi_v^* \Psi_v dx = \alpha \int_{-\infty}^{\infty} \Psi_v^* \Psi_v dy = \alpha \int_{-\infty}^{\infty} H_v^2(y) e^{-y^2} dy = \alpha \pi^{1/2} 2^v v!$$

v !=v(v-1)(v-2)…

1 u = 931,494 MeV/c^2

MATHEMATICS EQUATIONS

Integrals

$$\int_0^{\infty} \sqrt{x}\ e^{-x}\ dx = \frac{\sqrt{\pi}}{2}$$

$$\int_0^{\infty} e^{-ax^2}\ dx = \frac{1}{2}\sqrt{\frac{\pi}{a}}$$

$$\int_0^{\infty} x^2\ e^{-ax^2} dx = \frac{\sqrt{\pi}}{4a^{3/2}}$$

Law of cosines

$$c^2 = a^2 + b^2 - 2ab \cos \gamma$$

Stirling's approximation

$$n! \approx n^n e^{-n} \sqrt{2\pi n}$$

Identity

$$\nabla^2 \mathbf{A} = \nabla(\nabla \mathbf{A}) - \nabla \times \nabla \times \mathbf{A}$$

Trigonometric identities

$$\sin(\alpha \pm \beta) = \sin\alpha \cos\beta \pm \cos\alpha \sin\beta$$
$$\cos(\alpha \pm \beta) = \cos\alpha \cos\beta \mp \sin\alpha \sin\beta$$

$$\sin 2\theta = 2 \sin\theta \cos\theta$$
$$\cos 2\theta = \cos^2\theta - \sin^2\theta$$

$$\cos\alpha \cos\beta = \frac{1}{2}\left(\cos(\alpha+\beta) + \cos(\alpha-\beta)\right)$$

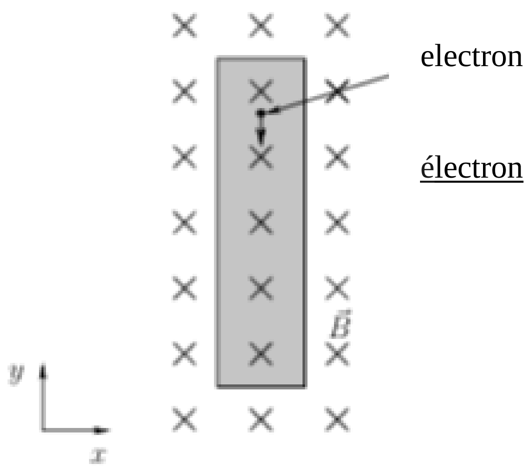
$$\sin\alpha \sin\beta = \frac{1}{2}\left(\cos(\alpha-\beta) - \cos(\alpha+\beta)\right)$$

$$\sin\alpha \cos\beta = \frac{1}{2}\left(\sin(\alpha+\beta) + \sin(\alpha-\beta)\right)$$

QUESTION 1 : **ELECTROMAGNETISM**

A crucial experiment that contributed to the establishment of the theory of electrical conduction in metals is based on a phenomenon related to magnetism and the ability of magnetic fields to interact with charges.

Consider the metal plate of the figure below (infinite length in the y direction), which is traversed by a current *I* in the negative direction of the y-axis whose electrons move with velocity v_d. Let *d* be the width of the plate (along the x axis) and δ its thickness.



1. **(5 points)** Indicate qualitatively - drawing a scheme - if and how the electrons generating the electric current *I* are diverted, if this deviation leads to the generation of an electric field and, if so, in which direction.

2. **(5 points)** After a certain time, a situation of equilibrium is established. Describe with the appropriate equations the balance of forces.

3. **(5 points)** Within the context of this problem, propose an approach to calculate the density of the electrons, *n*.

4. **(2 points)** Determine v_d and *n* for the following situation: d = 1 cm, δ = 10 µm, length 4 cm, I = 3 A, B = 1.5 T and V_H = 10 µV.

5. **(3 points)** Discuss the effect of the width of the plate with respect to the trajectory of the electron (as determined by the intensity of the magnetic field!), on the possibility to detect an electric field (see question 1).

***QUESTION* 2** : **QUANTUM MECHANICS**

A particle of mass *m* is under the influence of earth's gravitational field. We will consider 1D motion along the *z* axis and an infinite potential barrier below the earth's position (*z* = 0). The standard gravity constant is given by *g*.

a) **(5 points)** Write an expression for the Hamiltonian describing the particle's motion

consider the following wavefunction for the particle :

$$\psi(z) = z e^{-az}, \quad z > 0$$
$$0, \quad z \leq 0$$

where *a* is a constant parameter;

b) **(5 points)** Normalize this wavefunction;

c) **(5 points)** Calculate the average energy of the particle;

d) **(5 points)** Estimate the ground state energy by minimizing the expression found in c), with respect to the parameter *a*.

Possibly useful integrals :

$$\int x e^{ax}\, dx = e^{ax}\left(\frac{x}{a} - \frac{1}{a^2}\right)$$
$$\int x^2 e^{ax}\, dx = e^{ax}\left(\frac{x^2}{a} - \frac{2x}{a^2} + \frac{2}{a^3}\right)$$
$$\int x^3 e^{ax}\, dx = e^{ax}\left(\frac{x^3}{a} - \frac{3x^2}{a^2} + \frac{6x}{a^3} - \frac{6}{a^4}\right)$$

QUESTION 3 : **STATISTICAL PHYSICS**

A system is formed by *N* indistinguishable particles, which can occupy the following energy states: $\varepsilon_1 = +\varepsilon$ et $\varepsilon_2 = -\varepsilon$. If the volumetric mass of the system is low enough, consequently there is no interaction between the particles, you must:

a) **(8 points)** Develop an expression for the internal energy of the system as a function of its temperature;

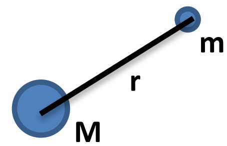
b) **(6 points)** Write down an equation that can be applied for calculating the heat capacity at constant volume of such a system;

c) **(6 points)** Analyse and explain the physical behavior of the heat capacity at constant volume as a function of the system's temperature.

Note: To carry out your calculations, you can use the following relation: $\cosh(x) = (e^x+e^{-x})/2$.

QUESTION 4 : **CLASSICAL MECHANICS**

Imagine a test particle of mass *m* orbiting a massive cosmological object of mass *M (M >> m)*.



(a) **[4 pts]** Assuming Newtonian mechanics and neglecting any motion of *M*, write down the conservation laws for energy (E) and angular momentum (L) in this system using polar coordinates (r,θ).

(b) **[3 pts]** Rewriting part (a) in the form

$$\frac{1}{2}m\dot{r}^2 = E - U_{eff}(r) \qquad \left(\dot{r} = \frac{dr}{dt}\right)$$

derive the formula for the effective potential $U_{eff}(r)$.

(c) **[3 pts]** Assuming the following form of $U_{eff}(r)$:

$$U_{eff}(r) = -\frac{GmM}{r} + \frac{L^2}{2mr^2} - \alpha\frac{L^2}{mr^3}$$

where α as an external parameter, derive the radius of a circular orbit for the Kepler's case ($\alpha = 0$).

(d) **[4 pts]** For the form of $U_{eff}(r)$ in part (c) with $\alpha = 1$, find a bound for angular momentum L, below which no circular orbits are possible. Show for which values of L there exist two circular orbits.

(e) **[3 pts]** Sketch a plot of $U_{eff}(r)$ when $\alpha = 1$ and for $L^2 < 12GMm^2$ as well as $L^2 > 12GMm^2$.

(f) **[3 pts]** Describe qualitatively the possible orbits in part (c).

QUESTION 5 : **OPTICS I**

A Fabry Perot cavity of length *L* is made of two mirrors with one 100% reflector and the other with a reflectivity of 98%. The cavity is entirely filled with a material that has optical gain which amplifies light propagating through it. Consider that the peak value of the optical gain G_0 = 5.164 x 10^{-4} cm^{-1} occurs at a wavelength 632.81nm and has a bandwidth (gain bandwidth, BW) of 5 pm. Assume the refractive index of the gain medium to be n = 1.00004.

1. **(5 points)** When light propagates in this cavity its amplitude grows according to $\boldsymbol{E_o \cdot exp(G_0 L)}$, where $\boldsymbol{E_0}$ is the amplitude of light before propagation through the gain medium. Determine the length of the cavity *L* at which the electric field after one cavity round-trip remains unchanged despite the loss at one of the mirrors.

2. **(5 points)** What is the mode spacing of this Fabry-Perot cavity in MHz?

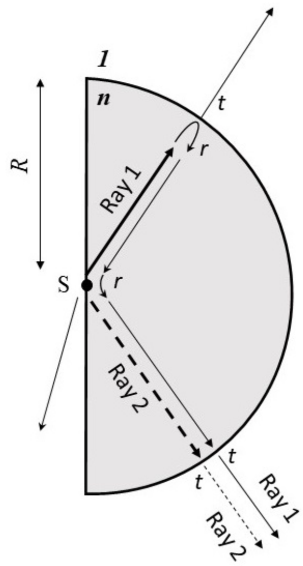
3. **(5 points)** With the cavity operating at equilibrium (i.e. the electric field amplitude in the cavity remains unchanged), how many cavity modes are contained within the gain bandwidth?

4. **(5 points)** The gain may be modeled as an inverse parabolic curve at the peak with the function:

$$g = G_0 - \left(\frac{\Delta\lambda}{\lambda_0}\right)^2$$

Determine at which exact wavelength there is a maximum power inside of the cavity.

QUESTION 6 : OPTICS 2



Wave optics. Multiple reflections in the semi-spherical lens.

The figure shows a semi-spherical lens of radius *R* and refractive index *n* placed in air with refractive index *1*. In the center of a semi-spherical lens, a multi-frequency source S of coherent radiation is placed that emits at all angles. The source is located inside of the lens, right next to its flat surface. When tracking the paths of the individual rays, one observes that some rays (say ray 1 in Figure) emitted by the point source S, can experience multiple reflections at the lens/air boundaries. This results in the interference of a direct ray (ray 2 in Figure) with the multiply reflected rays (for example, ray 1 in Figure, as well as other rays) that come out of the lens at the same angle. As a result, intensity of the outgoing beam will show spectral maxima or minima corresponding to the conditions of constructive or destructive interference of light inside of the lens.

Figure

a. **(5 points)** By summing up all the contributions from the direct and multiply reflected rays (in the lens) find the resultant frequency dependent complex electric field of the outgoing beam:

$$E(\nu) = E_0 \cdot t \cdot exp(i \cdot Rk_l) + E_0 \cdot t \cdot r^2 \cdot exp(i \cdot 3Rk_l) + \cdots ,$$

where *r* and *t* are the Fresnel reflection and transmission coefficients at the lens/air boundary, $k_l=2\pi\nu n/c$ is the light propagation constant (wavenumber) in the lens material, E_0 is the field amplitude of rays emitted by the source, and ν is the light frequency. For simplicity we consider that Fresnel reflection coefficient *r* is angle independent, purely real, and the same at all the lens/air boundaries. Also, we consider that Fresnel transmission coefficient *t* is purely real.

b. **(10 points)** Find an expression for the frequency dependent intensity of the outgoing beam:

$$I(\nu) = E(\nu) \cdot E^*(\nu).$$

Find expressions (in terms of the reflection and transmission coefficients) for the maximal $I_{max} = \max(I(\nu))$ and minimal $I_{min} = \min(I(\nu))$ values of the outgoing beam intensities, as well as corresponding frequencies ν_{max}, ν_{min} at which such values are achieved.

c. **(5 points)** Taking the following expressions for the Fresnel reflection and transmission coefficients at the lens/air boundaries:

$$t = \frac{2n}{n+1} \,; r = \frac{n-1}{n+1}$$

calculate the spectral modulation intensity of the outgoing beam I_{max}/I_{min} in terms of the lens and air refractive indices.

QUESTION 7 : **SOLID STATE PHYSICS I**

Suppose an intrinsic semiconductor whose electronic density of states function *N*(*E*) is depicted in Fig. 1.

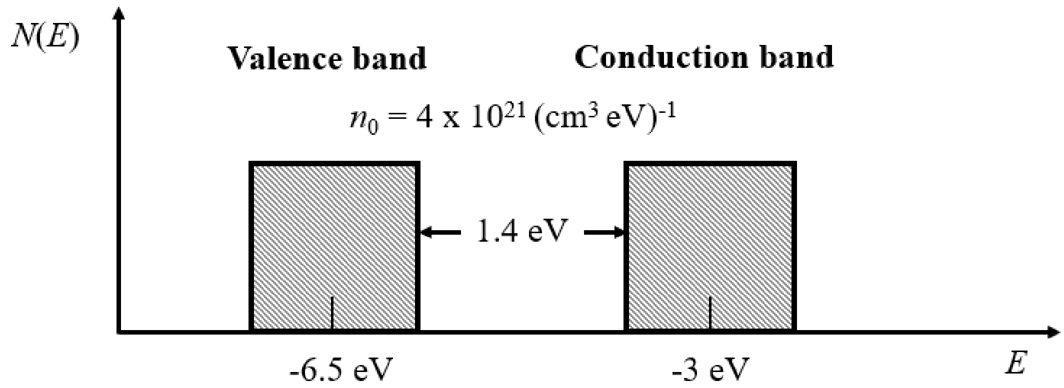


Figure 1. Electronic density of states function.

1. **(10 points)** Find expressions for *n* and *p* carrier concentrations assuming a Fermi-Dirac distribution function for occupation and that *N*(*E*) already contains the spin degeneracy (factor of 2).

   Note: You can assume that E_c - E_f and $E_f - E_v >> k_BT$, where E_c and E_v are the conduction and valence band edges

2. **(6 points)** Demonstrate and indicate where the Fermi level resides with respect to the valence and conduction bands?

3. **(4 points)** Estimate the density of conduction band electrons at room temperature.

QUESTION 8 : **SOLID STATE PHYSICS 2**

Ion chain vibrations
A one-dimensional chain is composed of N ions of mass m separated by an interatomic separation of a. The ion charges alternate signs from one ion to the next: *q*, *-q*, *q*, *-q*, ….

Hence, an analysis of the vibrational modes requires two terms. The first is the usual term describing a force proportional to displacements from equilibrium positions. This is the Hooke's law. The second term describes the electrostatic interactions between ions. It is described by Coulomb's law.

Only considering interactions **between nearest neighbors**,

a) **(5 points)** Write the equation of motion for the ion located at position n, where n is an index representing an ion on this chain. Use $u_n(t)$, the displacement of ion n from its equilibrium position, and assume that this displacement is small with respect to the interatomic separation. Hence, only consider linear terms in the equation of motion.

b) **(5 points)** Without solving the equation found in a), give the mathematical form of $u_n(t)$ representing the **normal vibrational mode**, as a function of time *t* and wave number *k,* resulting from the following periodic condition $u_n(t) = u_{n+N}(t)$.

c) **(5 points)** Using $u_n(t)$ as a solution of the equation of motion found in a), give the dispersion relation $\omega(k)$.

d) **(5 points)** Determine the sound velocity. Discuss of the effect of the iconicity of the chain on the phonon frequency and the sound velocity.