POLYTECHNIQUE MONTRÉAL
LE GÉNIE EN PREMIÈRE CLASSE

EXAMEN GÉNÉRAL DE SYNTHÈSE – ÉPREUVE ÉCRITE
Programme de doctorat en génie physique

Jeudi 18 juin 2015

Salle B-508

de 9h30 à 13h30

NOTES :

- *No documentation allowed.*
- *A non-programmable calculator is allowed.*
- *The candidate can answer up to 6 questions of his choice.*
- *Each question is worth 20 points.*
- ***Use a notebook different for each question, making sure to include the question number on it.***
- *This questionnaire contains 8 questions, 10 pages.*

ENGLISH VERSION

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CONSTANTES PHYSIQUES :

$e = 1.602 \times 10^{-19}$ C

$m_e = 9{,}109 \times 10^{-31}$ Kg

$\hbar = 1.055 \times 10 \times 10^{-34}$ J·s

$k_B = 1.381 \times 10^{-23}$ J/K

$\varepsilon_0 = 8.854 \times 10^{-12}$ F/M

$\mu_0 = 4\pi \times 10^{-7}$ N/A^2

ÉQUATIONS PHYSIQUES :

$$\nabla \cdot \mathbf{D} = \rho_f$$

$$\nabla \cdot \mathbf{B} = 0$$

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$$

$$\nabla \times \mathbf{H} = \mathbf{J}_f + \frac{\partial \mathbf{D}}{\partial t}$$

$$\mathbf{B} = \frac{\mu_0}{4\pi} \int \frac{\mathbf{J} \times \hat{\mathbf{r}}}{|\mathbf{r}|^2}\, dV$$

$$f_{FD}(E) = \frac{1}{e^{(E-\mu)/k_B T} + 1}$$

$$f_{BE}(E) = \frac{1}{e^{(E-\mu)/k_B T} - 1}$$

Clausius-Mossotti relation

$$\frac{n\alpha}{3\varepsilon_0} = \frac{\varepsilon_r - 1}{\varepsilon_r + 2}$$

ÉQUATIONS MATHÉMATIQUES

Intégrale

$$\int_0^\infty \sqrt{x}\, e^{-x}\, dx = \frac{\sqrt{\pi}}{2}$$

$$\int_0^\infty e^{-ax^2}\, dx = \frac{1}{2}\sqrt{\frac{\pi}{a}}$$

$$\int_0^\infty x^2 e^{-ax^2} dx = \frac{\sqrt{\pi}}{4a^{3/2}}$$

Loi des cosinus

$$c^2 = a^2 + b^2 - 2ab\cos\gamma$$

Approximation de Stirling

$$n! \approx n^n e^{-n} \sqrt{2\pi n}$$

Identité

$$\nabla^2 \mathbf{A} = \nabla(\nabla \mathbf{A}) - \nabla \times \nabla \times \mathbf{A}$$

Identités trigonométriques

$$sin(\alpha \pm \beta) = sin\,\alpha\, cos\,\beta \pm cos\,\alpha\, sin\,\beta$$
$$cos(\alpha \pm \beta) = cos\,\alpha\, cos\,\beta \mp sin\,\alpha\, sin\,\beta$$

$$\sin 2\theta = 2\sin\theta\cos\theta$$
$$\cos 2\theta = \cos^2\theta - \sin^2\theta$$

$$\cos\alpha\cos\beta = \frac{1}{2}\left(\cos(\alpha+\beta) + \cos(\alpha-\beta)\right)$$

$$\sin\alpha\sin\beta = \frac{1}{2}\left(\cos(\alpha-\beta) - \cos(\alpha+\beta)\right)$$

$$\sin\alpha\cos\beta = \frac{1}{2}\left(\sin(\alpha+\beta) + \sin(\alpha-\beta)\right)$$

QUESTION 1 : **ELECTRICITY AND MAGNETISM**

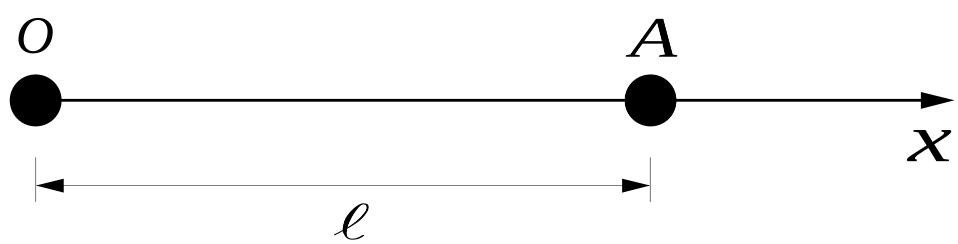
We consider two conductive point-balls located at *O* and A along the $O\hat{x}$ axis, where the distance between *O* and *A, d(OA) = l (l > 0)*. The charge at *O* is *Q > 0* and the one at *A* is *Q=n (n > 0)*. We then introduce a third point-ball *M* that is geometrically identical to the previous ones but which can move freely on the $O\hat{x}$ axis between *O* and *A*. The point-ball *M* initially neutral, is brought into contact with the point-ball at *O*.

a) **(2 points)** Knowing that for an isolated system the charge is conserved, what are the charge on *M* and *O* after the contact?

b) **(6 points)** Express the equilibrium distance x_e between *O* and *M* in terms of *l* and *n*.

c) **(8 points)** What is the direction of the force on *M* when it is displaced from its equilibrium position by a very small distance *dx*? Is it a stable equilibrium? Explain.

d) **(4 points)** For what value of *n* do we have x_e= *3l/4*?



***QUESTION* 2** : QUANTUM MECHANICS

Consider a monodimensional harmonic oscillator constituted of a particle with mass *m* and characterized by a restitution force constant *k*. For this harmonic oscillator:

$$\langle x \rangle = 0$$

$$\langle x^2 \rangle = (v + \frac{1}{2}) \frac{\hbar}{(mk)^{\frac{1}{2}}} \quad \text{with } v = 0, 1, 2 \ldots$$

1) **(4 points)** Indicate the normalized wavefunction that describes the ground state of the harmonic oscillator.

2) **(4 points)** Verify that the wavefunction in 1) is a solution of the Schröedinger equation describing the oscillator.

3) **(4 points)** Indicate the mean kinetic energy and the mean potential energy for this oscillator. Compare the mean values of the two forms of energy.

4) **(4 points)** Calculate the smallest value of the energy permitted by the uncertainty principle for this oscillator.

Reminder : $\Delta p = \left\{ \langle p^2 \rangle - \langle p \rangle^2 \right\}^{1/2} \qquad \Delta x = \left\{ \langle x^2 \rangle - \langle x \rangle^2 \right\}^{1/2}$

5) **(4 points)** Evaluate if the model of the harmonic oscillator properly describes the vibrational behavior of a molecule? Discuss the answer.

QUESTION 3 : **STATISTICAL PHYSICS**

Concentration of positron in a Tokamak at high temperatures

From the point of view of statistical physics, the Tokamak is a box of volume V that contains a plasma heating its walls at a temperature T. These walls are made of atoms that can absorb and emit photons. The number of photons and their energy inside the Tokamak is therefore fluctuating. In addition, the pair production reaction

$$\gamma \rightarrow e_- + e_+$$

is possible when the photon energy $\varepsilon_\gamma > 2m_e c^2$. Here, $m_e = 0.511$ MeV/c^2 is the mass of an electron (e_-) or positron (e_+) and c is the speed of light. Thus, positrons (i.e. anti-electrons) will always be present in the Tokamak.

For this problem, the chemical potentials of photons, electrons and positrons are zero.

a) **(10 points)** Give a relation for $n_\gamma(T) = N_\gamma/V$ the concentration of photons in the Tokamak. The energy of a photon is given by $\varepsilon_\gamma = p_\gamma c$ with p_γ the momentum of the photon.

b) **(10 points)** What is the concentration $n_p(T) = n_e(T) = N_e/V$ of positrons in the same Tokamak resulting from the pair creation reaction? The total energy of an electron (non relativistic approximation) is

$$\varepsilon_e = m_e c^2 + \frac{p_e^2}{2m_e}$$

with p_e the momentum of the electron. You can assume that the temperature of the Tokamak is very high but much smaller than $m_e c^2/k_B$ ($k_B = 8.62 \times 10^{-5}$ eV/K is the Boltzmann constant).

The following integrals might be useful:

$$\int_0^\infty \frac{x^2}{e^x - 1} dx \approx 2.40411$$

QUESTION 4 : **CLASSICAL MECHANICS**

A physics engineer has designed a Laser-Induced Breakdown Spectroscopy (LIBS) instrument similar to the one used by NASA in the *Curiosity Mars Rover*[1] program. This instrument necessitates the linear displacement of an optical lens. To this purpose, the engineer has proposed to use the mechanical set-up schematized in the Figure below. By knowing that the focus (i.e. the focal point *f* in the figure) must move along the *x-axis* with a given periodicity, in order to help the engineer you must:

a) **(6 points)** Find the equation of the velocity at which the lens will move as a function of a pulsation frequency Ω and the angular positions φ and θ;

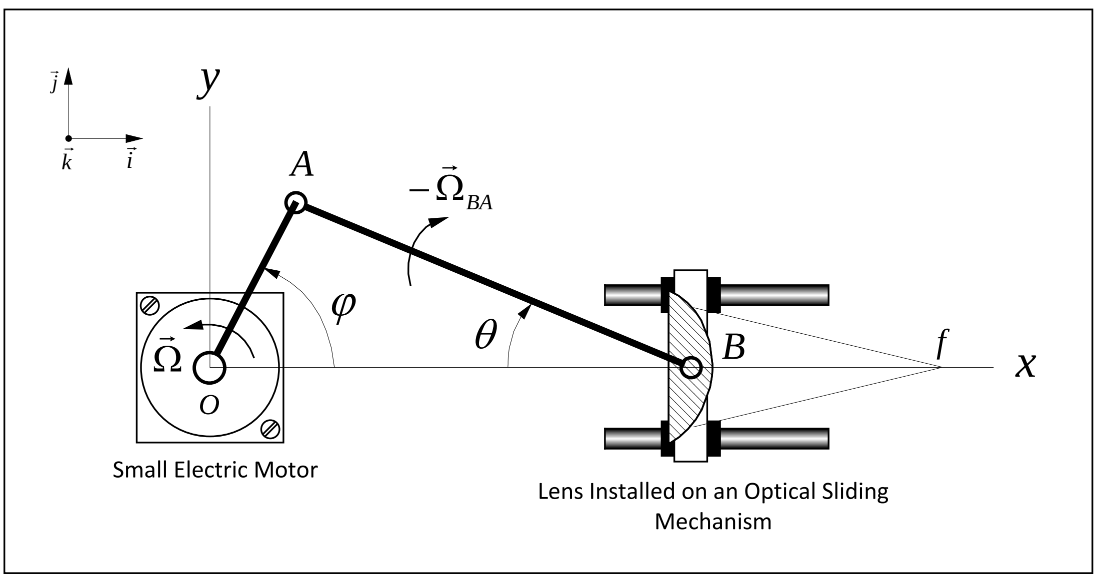
b) **(6 points)** Demonstrate that for a parameter $\lambda = \frac{\overline{OA}}{\overline{BA}} < 1$, the velocity of the lens can be approximated by the following relation:

$$\dot{x}(\Omega\tau) = V_B \approx -\overline{OA}\,\Omega\sin(\Omega\tau)[1-\lambda\cos(\Omega\tau)];$$

c) **(3 points)** Assuming $\lambda \approx 0$, determine the maximum force excreted in the direction of the displacement by the lens displacement system which has an overall mass *m*;

d) **(3 points)** Determine the torque at the center of the motor for the same condition applied in item c);

e) **(2 points)** Estimate the maximum force in the direction of the displacement when the motor turns at 3000 *RPM* with $\overline{OA} = 0.5\,cm$ and *m* = 50 *gr* (You must neglect the effect of friction as well as the inertia of the mechanical connectors *O-A* and *B-A*).



Proposed mechanism for a LIBS moving lens

[1] NASA Web site: http://mars.jpl.nasa.gov/msl/mission/instruments/spectrometers/.

QUESTION 5 : **OPTICS I**

WAVE OPTICS

A Michelson interferometer is used to measure the deflection of a mobile arm on which a mirror is mounted. The interferometer is shown on Figure 1. The optical path is equal in both branches of the interferometer when the moving arm is at its center position *($\theta = 0$)*. The rotation of the moving arm causes both a rotation of the mirror and a translation of its center. The beam-splitter has transmission and reflection coefficients of 50% and has negligible thickness.

The incoming beam of light is a uniform monochromatic plane wave of wavelength λ and intensity I_0. At the output of the interferometer is a screen on which the intensity distribution can be observed.

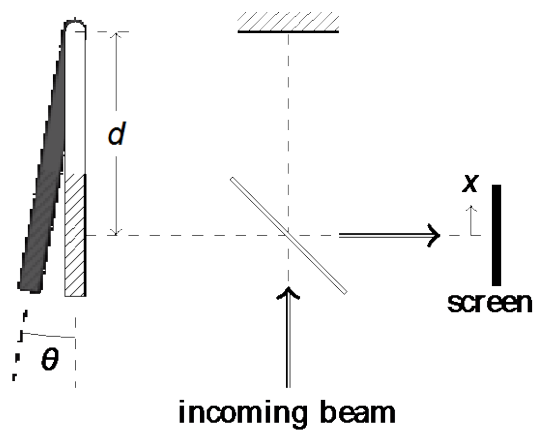


Figure 1 — Michelson interferometer containing a moving mirror.

Answer the following questions.

1. **(8 points)** Calculate the intensity at the center of the screen *($x = 0$)* as a function of the rotation θ of the mirror;

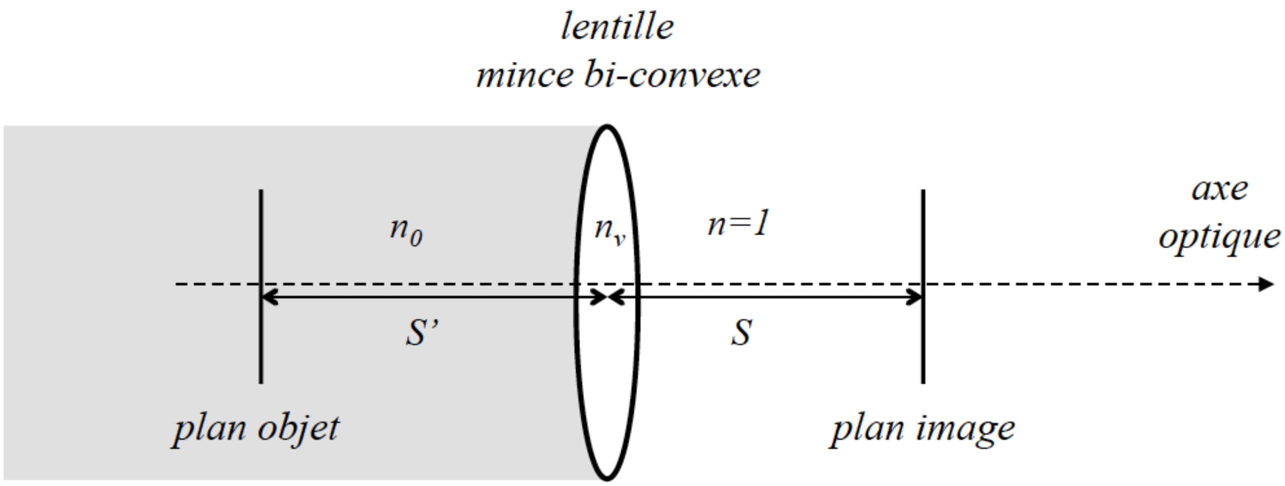
2. **(8 points)** What is observed on the screen when the mirror is rotated? Make a quantitative description as a function of the rotation θ;

3. **(4 points)** The monochromatic source is now replaced by a plane uniform white light source, therefore incoherent. Describe qualitatively what is observed on the screen as a function of the rotation.

QUESTION 6 : **OPTICS 2**

BI-CONVEX THIN LENS

A bi-convex thin lens with the same radius of curvature *R* (absolute value) on both sides is used in the imaging system illustrated below. The object plane is in a n_0 refractive index medium, the index of the glass of the lens is n_v and the image plane is in the air (n = 1).



Translation : 'plan objet' = 'object plane', 'plan image' = 'image plane', 'axe optique' = 'optical axis', 'lentille mince bi-convexe' = 'bi-convex thin lens'.

(a) **(7 points)** Find the transfer matrix associated with the system to model the propagation of rays from the object plane to the image plane;

(b) **(7 points)** Using the matrix elements discovered in (a) you must find: (i) the relationship between *S* and *S'*, and (ii) the magnification;

(c) **(6 points)** Compare imaging conditions (position of the images , focal lengths , magnification) with a system for which n_0 = n = 1 .

QUESTION 7 : **SOLID STATE PHYSICS I**

Dielectric properties of water in different states

a) **(8 points)** Name three mechanisms of dielectric polarization of water (H_2O). Give the approximate frequency range over which these mechanisms are effective. Discuss whether or not they are effective when the water is (i) gaseous, (ii) liquid and (iii) solid.

b) **(4 points)** Based on your answers in a), explain your qualitative expectations for the optical index of refraction in these three states of water (H_2O). In other words, do you expect the index of refraction to be relatively higher, lower or roughly equal in these three states, and why?

c) **(4 points)** Same question as b), but now we are interested in the low frequency dielectric constant (suppose the material is used as the dielectric in a capacitor). Do you expect the dielectric constant to be higher, lower or roughly equal in these three states of matter, and why?

d) **(4 points)** In calculations of the dielectric constant or index of refraction from the microscopic polarizability of atoms and molecules, we have to use a local electric field, which is different from the average macroscopic internal field (appearing in Maxwell's equation), which is also distinct form the external applied field. Explain the difference between these three fields.

QUESTION 8 : **SOLID STATE PHYSICS 2**

Consider a very long line of *2N ions (N>>1)* of alternating charges ±q with a repulsive potential A/R^n between nearest neighbors in addition to the usual Coulomb potential. Note that A is a constant, R the distance between nearest neighbors and $n>1$.

a) **(7 points)** Show that the total Coulomb potential is given by

$$U(R) = -\frac{N a q^2}{R}$$

Where a is the Madelung constant. Show that $a \approx 1{,}386$.

b) **(6 points)** Find the equilibrium separation R_o for such a system and calculate the equilibrium energy $U(R_o)$.

c) **(7 points)** Let the crystal be compressed so that R_o becomes $R_o(1-\delta)$ where $\delta < 1$. Calculate the work W done in compressing a unit length of the crystal of the system. Show that W is proportional to δ^2.