POLYTECHNIQUE MONTRÉAL

TECHNOLOGICAL UNIVERSITY

QUALIFICATION EXAM – WRITTEN PART
Ph. D. Program in Engineering Physics

<u>Thursday November 18, 2021</u>

Room C-539.4

from 9h30 to 13h30

<u>NOTES</u> :

- No documentation is allowed.
- A non-programmable calculator is allowed.
- A page of mathematical and physics equation is provided at page 2.
- This examination contains 8 questions, 10 pages in total.
- Each question is worth 20 points.
- Provide solutions to no more than 6 questions of your choice.
- Use a different notebook for each question, clearly making each notebook with the corresponding question number.

ENGLISH VERSION

Department Engineering Physics
Main Building
Téléphone : 514-340-4787
Télécopieur : 514-340-3218
Courriel : info@phys.polymtl.ca

Adresse postale
P.O. Box 6079, Station Centre-ville
Montréal (Québec) Canada H3C 3A7
www.polymtl.ca

2900, boul. Édouard-Montpetit
Campus of the'Université of Montréal
2500, chemin de Polytechnique
Montréal (Québec) Canada H3T 1J4

FORMULAS AND USEFUL RELATIONS

Constants

$1\ \text{Å} = 10^{-10}\text{m}$	$1\ \text{eV} = 1.602 \times 10^{-19}\ \text{J}$	$h = 6.626 \times 10^{-34}\ \text{J s}$
$c = 2.998 \times 10^{8}\ \text{m/s}$	$k_B = 1.381 \times 10^{-23}\ \text{J/K}$	$\epsilon_0 = 8.854 \times 10^{-12}\ \text{C}^2/\text{N m}^2$
$m_e = 9.109 \times 10^{-31}\ \text{kg}$	$m_p = 1.672 \times 10^{-27}\ \text{kg}$	$\lvert e\rvert = 1.602 \times 10^{-19}\ \text{C}$
$N_A = 6.023 \times 10^{23}\ \text{mol}^{-1}$	$a_0 = 0.5291 \times 10^{-10}\ \text{m}$	$\mu_0 = 4\pi \times 10^{-7}\ \text{N}^2/\text{A}^2$
	$\mu_B = 5.79 \times 10^{-5}\ \text{eV T}^{-1}$	

Physics equations

$$\nabla \cdot \mathbf{D} = \rho_f \ ; \qquad \nabla \cdot \mathbf{B} = 0 \ ; \qquad \nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \ ; \qquad \nabla \times \mathbf{H} = \mathbf{J}_f + \frac{\partial \mathbf{D}}{\partial t} \ ; \qquad \mathbf{B} = \frac{\mu_0}{4\pi} \int \frac{\mathbf{J} \times \hat{\mathbf{r}}}{|\mathbf{r}|^2} dV$$

$$f_{\mathrm{FD}}(E) = \frac{1}{e^{(E-\mu)/k_B T} + 1} \ \text{(distrib. Fermi-Dirac)}; \qquad f_{\mathrm{BE}}(E) = \frac{1}{e^{(E-\mu)/k_B T} - 1} \ \text{(distrib. Bose-Einsten)}$$

$$\frac{N\alpha}{2\varepsilon_0} = \frac{\varepsilon_r - 1}{\varepsilon_r + 2} \ \text{(Clausius-Mossotti relation)}; \qquad E_n = \frac{n^2 \pi^2 \hbar^2}{2mL^2} \ \text{(energy of an infinite 1D well)}$$

Integrals

$$\int \frac{dx}{e^x + 1} = x - \ln(e^x + 1) \qquad ; \qquad \int \frac{dx}{e^x - 1} = \ln(e^x - 1) - x$$

$$\int_0^\infty x^n e^{-qx} dx = \frac{n!}{q^{n+1}}, \quad n > -1, q > 0 \ ; \qquad \int_0^\infty e^{-\alpha^2 x^2} dx = \frac{1}{2\alpha}\sqrt{\pi}$$

$$\int_0^\infty x^{2n} e^{-\alpha x^2} dx = \frac{1 \cdot 3 \cdot 5 \cdots (2n-1)}{2^{n+1}\alpha^n}\sqrt{\frac{\pi}{\alpha}} \ ; \qquad \int_0^\infty x^{2n+1} e^{-\alpha x^2} dx = \frac{n!}{2\alpha^{n+1}}, \quad (\alpha > 0)$$

Trigonometric identities

$$\cos(2a) = \cos^2 a - \sin^2 a = 2\cos^2 a - 1 = 1 - 2\sin^2 a \ ; \qquad \sin(2a) = 2 \sin a \cos a$$

$$\sin a \sin b = \frac{1}{2}\cos(a-b) - \frac{1}{2}\cos(a+b) \ ; \qquad \cos a \cos b = \frac{1}{2}\cos(a-b) + \frac{1}{2}\cos(a+b)$$

$$\sin a \cos b = \frac{1}{2}\sin(a+b) + \frac{1}{2}\sin(a-b) \ ; \qquad \cos a \sin b = \frac{1}{2}\sin(a+b) - \frac{1}{2}\sin(a-b)$$

$$\cos a + i \sin a = e^{ia} \ ; \qquad \cos a - i \sin a = e^{-ia}$$

Others

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \dots \ ; \qquad (1+x)^n = 1 + nx + \frac{n(n-1)}{2!}x^2 + \dots$$

$$c^2 = a^2 + b^2 - 2ab\cos\gamma \ \text{(cosine law)} \ ; \qquad n! \approx n^n e^{-n}\sqrt{2\pi n} \ \text{(Stirling approximation)}$$

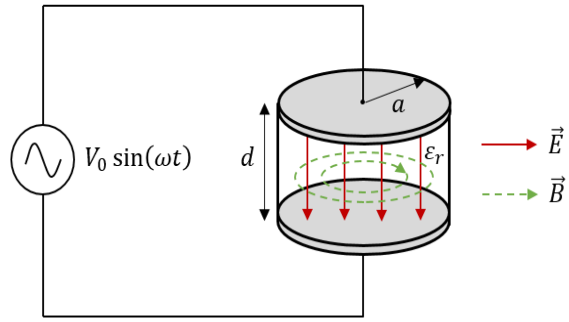
$$\nabla^2 \mathbf{A} = \nabla(\nabla \mathbf{A}) - \nabla \times \nabla \times \mathbf{A} \ \text{(Identity)} \ ; \qquad \sum_{n=0}^{\infty} x^n \approx \frac{1}{1-x}, \quad |x| < 1 \ \text{(Serie)}$$

$$\sqrt{1 \pm x^2} \approx 1 \pm \frac{x^2}{2}, \quad \text{if } x \ll 1$$

QUESTION 1 : **ELECTROMAGNETISM**

Parasitic inductance of a capacitor

In this problem, we aim to study high-frequency inductive effects in a simple circuit formed by a parallel-plate capacitor driven by a sinusoidal AC voltage source $V(t) = V_0 \sin(\omega t)$.



The plates of the capacitor consist of two disks of radius a and negligible thickness. The material of the plates themselves is a perfect conductor, while the material between the plates is a lossless dielectric of relative permittivity ε_r.

We expect to see inductive effects because variations in the amount of charge on the plates (caused by the AC voltage) will induce time-varying electromagnetic fields $\vec{E}$ and $\vec{B}$ between the plates.

We make the assumption that fringe effects are negligible.

a) **(6 pts)** Using Maxwell's equations, find the expression of the capacity C of the capacitor.

b) **(6 pts)** Using Maxwell's equations, find the expression of the magnetic field $\vec{B}$ between the plates of the capacitor.

c) **(6 pts)** From the expression of the energy stored in the magnetic field between the plates of the capacitor, show that the parasitic inductance L of the capacitor is given by :

$$L = \mu_0 d / 8\pi.$$

d) **(2 pts)** How does this inductance impact the dynamical response of the circuit ?

Universal constants

$e = 1.602 \times 10^{-19}$ C $\qquad \varepsilon_0 = 8.854 \times 10^{-12}$ F/m $\qquad \mu_0 = 4\pi \times 10^{-7}$ H/m

Capacity

$$C = Q/V$$

Inductance

$$L = N\Phi/I$$

Energy stored in the electromagnetic field

$$U_E = \int \frac{1}{2} \varepsilon E^2 d^3r = \frac{1}{2} C V^2 \qquad U_B = \int \frac{1}{2\mu} B^2 d^3r = \frac{1}{2} L I^2$$

Maxwell's equations

$$\int_S \vec{D} \cdot d\vec{S} = Q \qquad \int_S \vec{B} \cdot d\vec{S} = 0$$

$$\oint_C \vec{B} \cdot d\vec{r} = \int_S \left(\vec{J} + \vec{J}_D\right) \cdot d\vec{S} \qquad \oint_C \vec{E} \cdot d\vec{r} = -\frac{d}{dt} \int_S \vec{B} \cdot d\vec{S}$$

QUESTION 2 : **QUANTUM MECHANICS**

Consider a well of semi-infinite potential defined by:

$$V(x) = \begin{cases} \infty & -\infty < x < 0 \\ 0 & 0 < x < L \\ U & L < x < +\infty \end{cases}$$

where U > 0

a) **(12 pts)** Show that for E < U, the solution of the time-independent Schrödinger equation is given by

$$\varphi(x) = \begin{cases} 0 & -\infty < x < 0 \\ A\sin(kx) & 0 < x < L \\ A\sin(kL)e^{-\alpha(x-L)} & L < x < +\infty \end{cases}$$

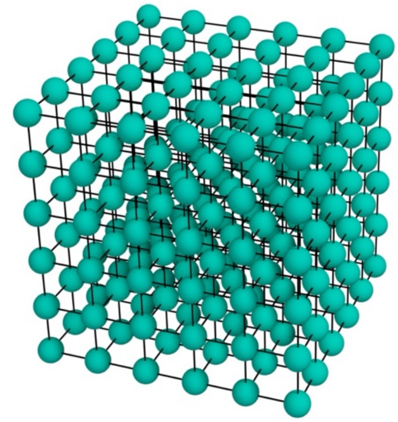
where

$k = \frac{\sqrt{2mE}}{\hbar}$, $\alpha = \frac{\sqrt{2m(U-E)}}{\hbar}$

b) **(4 pts)** Give an expression allowing to find the energies of the bound states.

c) **(4 pts)** Find an expression for A.

QUESTION 3 : **STATISTICAL PHYSICS**



Consider a crystal of N atoms as shown in the figure opposite. Point defects can occur, for example, when atoms are shifted from their regular lattice sites into the spaces between the atoms.
The atom in the interstitial space and the resulting gap are called a Frenkel pair. A lattice has n (with $1 \ll n \ll$ N) Frenkel pairs. The energy for the production of a Frenkel pair is ε_0. N' is the number of interstitial spaces.

Let the system be in thermal equilibrium for temperature T.

a) (4 pts) The microcanonical partition function is

$$\Omega(E) = \frac{N!}{n!\,(N-n)!} \cdot \frac{N'!}{n!\,(N'-n)!}$$

Describe in two sentences the meaning of the two multipliers for the microcanonical partition function.

b) (8 pts) Calculate the free energy of the crystal

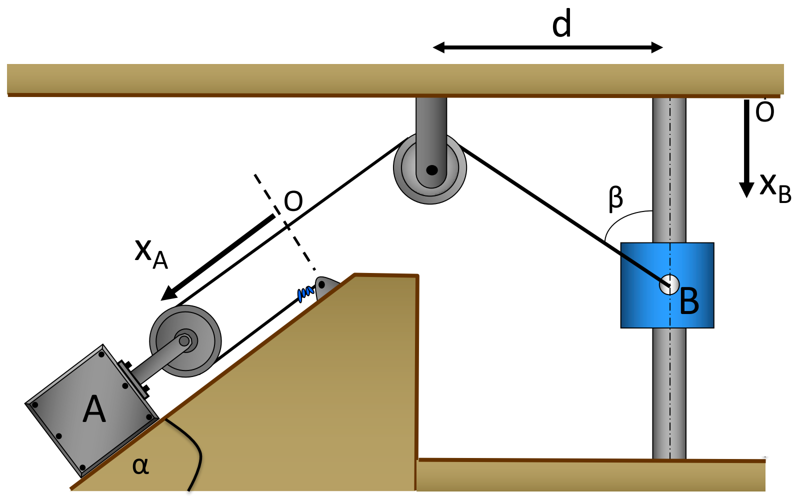
$$F = \Delta E - TS$$

Remember that the energy for the production of a Frenkel pair is ε_0.
First calculate the entropy $S = k_B \ln \Omega(\mathrm{E})$.
Considering that $1 \ll n \ll N, N'$, Stirling's formula may be of help

c) (8 pts) In thermal equilibrium, the free energy should be minimal.
Show that for $\varepsilon_0 \gg k_B T$

$$\frac{n^2}{(N-n)(N'-n)} = e^{-\frac{\varepsilon_0 \cdot}{\mathrm{k_B T}}}$$

QUESTION 4 : CLASSICAL MECHANICS



We let the mechanical system described in the above figure go from rest. The origin and the orientation of a set of generalized coordinates is indicated on the figure. The two masses M_A and M_B weigh each 10 kg. The two pulleys are massless. The rope is inextensible. The angles α and β (initial value) are respectively of 45 degrees each. Neglect any friction.

a) **(4 pts)** What is the relationship between the infinitesimal displacements dx_A et dx_B for an arbitrary position of A and B as a function of x_A and x_B only. (Note that the angle β depends on x_B).

b) **(4 pts)** In what direction will the mass M_B move. Up or down? Prove your answer.

c) **(2 pts)** If we wanted to equilibrate the system in the position shown above, how much mass should we add or remove to mass B to achieve static equilibrium?

d) **(10 pts)** By using a lagrangian approach, give the equation of movement of mass B as a function of variable x_B and the constants of the system only.

QUESTION 5 : **GEOMETRICAL OPTICS**

An optical system may be described by an ABCD matrix such that:

$$\begin{bmatrix} y_2 \\ \theta_2 \end{bmatrix} = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} y_1 \\ \theta_1 \end{bmatrix}, \tag{1}$$

where $y_{1,2}$ represents the heights of a ray entering and exiting the system, and $\theta_{1,2}$ represents the corresponding angles with respect to the optical axis. **Convention:** the label 1 refers to a ray entering the optical system, and the label 2 refers to the same ray as it exits the system.

The ABCD matrix may be used to characterize the 4f system that can be used for optical filtering applications to produce a filtered image in the plane E_i associated with an object in plane E_0 (Figure 1).

The system is composed of two thin lenses L_1 and L_2 having focal lengths f_1 et f_2, respectively. The lenses are separated by a distance $f_1 + f_2$, while the object and image planes are located at distances f_1 and f_2 from L_1 and L_2, respectively. The aperture *P (x, y)* is of variable diameter and used to filter high spatial frequencies during the image formation process.

The total distance between the object and image planes is 4f if f_1 = f_2. The red and blue lines of Figure 1 show exemplary rays for on- and off-axis conjugation, respectively.

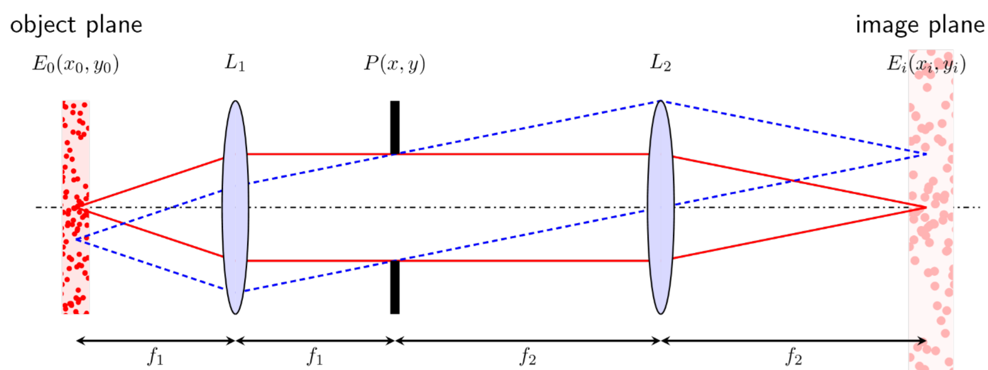


Figure 1 Schematic diagram of an afocal telescope, also called 4f-system, and showing two lenses along with the object and image planes.

a) (**10 pts**) Using the convention in Equation 1, derive the ABCD matrix for the 4f system from the object plane to the image plane. **For this question you must ignore the presence of the aperture *P(x,y)*.**

b) (**6 pts**) By inspection of each matrix component (A, B, C, D) associated with the system, one may deduce its imaging characteristics, including -for example- lateral magnification and angular magnification.
Using the matrix elements computed in a), describe the physical significance of each matrix element and justify your answer mathematically. **Hint: You can answer the question by describing what happens when each matrix element is made to be equal to 0.**

c) (**4 pts**) Describe how reinserting the aperture *P(x,y)* would affect the computation of the matrix elements.

QUESTION 6 : **WAVES OPTICS**

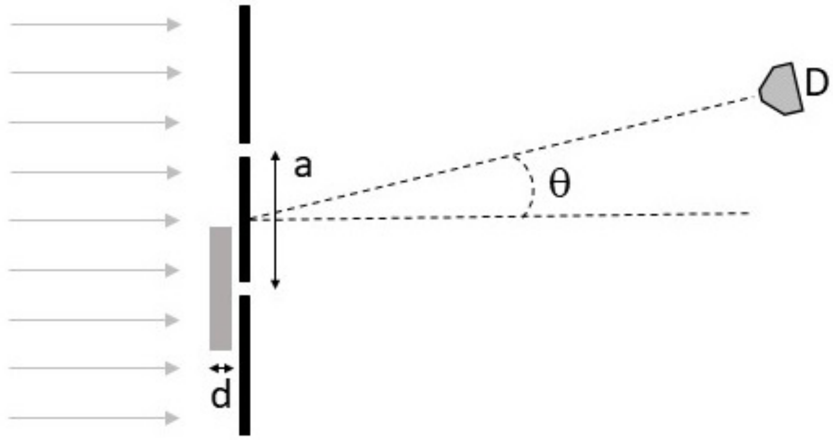


Fig 1. Scattering of the parallel beam on two slits and a phase plate.

A parallel beam of monochromatic light is incident onto an opaque screen with air on both sides. The screen features two small slits that are separated by distance **a**. One slit is open to air of refractive index 1, while another slit can be either opened to air or covered by a phase plate of thickness **d** and refractive index **n**>1. A detector D that measures light intensity is placed at the shallow elevation angle $\theta \ll 1$ from the system center line as shown in Fig. 1. Assuming that the incident light is generated by a tunable source of monochromatic light that scans over all frequencies from zero to infinity, the detector will then register a sequence of intensity minima and maxima.

a) **(5 pts)** First assume that there is NO phase plate covering any slits. Then, find frequencies at which detector will register intensity maxima and minima.
b) **(7 pts)** Next, assume that the phase plate is added to a setup covering only the lower slit. Find the new frequencies at which detector will register intensity maxima and minima.
c) **(8 pts)** The following procedure can then be used for detection of changes in the material refractive index of a phase plate. Particularly, assume operation at a fixed frequency ν chosen so that the detector registers a maximum of intensity for the unperturbed value of the phase plate refractive index **n**. Consider a small change in the refractive index of the phase plate $\delta \mathbf{n}>0$ so that the detector changes its reading from a maximum to a minimum. The smallest value of $\delta \mathbf{n}$ that changes the detector reading from the maximum to a minimum can be considered as the sensor resolution. Find the sensor resolution.

When solving the problem use the following simplifications:

- *Consider the distance between the detector and the screen to be much larger than both the slit size and separation between the slits.*
- *The slits are so small that they can be considered as point sources of the secondary waves.*
- *Disregard back reflections and multiple reflections in the phase plate.*
- *Consider detector size as negligible (a point detector).*

QUESTION 7 : **SOLID STATE PHYSICS 1**

Consider a two-dimensional square lattice of lattice constant *a*, such that $\hbar^2/(ma^2) = 1$ eV. The relations giving the band structure for the first band (ε_1) and the second band (ε_2) are

$$\varepsilon_1(k_x, k_y) = \frac{\hbar^2}{ma^2}\left(2 - \cos k_x a - \cos k_y a\right)$$

et

$$\varepsilon_2(k_x, k_y) = \frac{\hbar^2}{ma^2}\left(10 - 2\cos k_x a - 2\cos k_y a\right)$$

a) **(8 pts)** Draw $\varepsilon(\underline{k})$ in the directions $M \rightarrow \Gamma \rightarrow X$. Indicate the values of k_x and k_y at the points M, Γ et X.

b) **(4 pts)** If ε_F = 5 eV, determine if this material is a semiconductor, a metal or an insulator. What is the width of the forbidden band? Is this band gap direct or indirect?

c) **(8 pts)** Calculate the effective masses m^*_{xx} et m^*_{yy} (i) of the holes at the top of the valence band and (ii) of the electrons at the bottom of the conduction band.

QUESTION 8 : **SOLID STATE PHYSICS 2**

Sodium (Na) is a simple monovalent metal with a face centered cubic (FCC) structure, with a lattice constant of $a = 428$ pm (1 pm = 10^{-12} m). Its density is 0.971 g/cm^3 and its atomic mass is 23.0 g/mol. Let us assume that its paramagnetic properties are well described using the free electron gaz model at zero temperature.

a) **(2 pts)** Calculate the electronic density (electrons/m^3) of Na.

b) **(8 pts)** Determine the expression and the numerical value of the Fermi energy.

c) **(3 pts)** Calculate the magnetic field required to produce a relative shift of the spins band by 1% of the Fermi level.

d) **(5 pts)** Calculate the value of the magnetic moment, induced by this magnetic field, in μ_B/atom.)

e) **(2 pts)** Briefly discuss the feasibility to produce such a magnetic field in a laboratory.

Reminder – Density of states of a free electron gaz in 3D : $g(\varepsilon_F) = 3N/2\varepsilon_F$

Constants:

$m_e = 9.11\times10^{-31}$ kg	$\hbar = 1.055\times10^{-34}\,\mathrm{Js}$	$1\,\mathrm{eV} = 1.60\times10^{-19}\,\mathrm{J}$
$N_A = 6.02\times10^{23}\,\mathrm{mol}^{-1}$	$\mu_B = 5.79\times10^{-5}\,\mathrm{eV\,T}^{-1}$	$\mu_0 = 4\pi\times10^{-7}$