POLYTECHNIQUE MONTRÉAL
TECHNOLOGICAL UNIVERSITY

QUALIFICATION EXAM – WRITTEN PART
Ph. D. Program in Engineering Physics

<u>Thursday November 21, 2019</u>

Room C-539.4 (Main Building)

from 9h30 to 13h30

<u>NOTES</u> :

- *No documentation is allowed.*
- *A non-programmable calculator is allowed.*
- *This examination contains 8 questions, 10 pages in total.*
- *Each question is worth 20 points.*
- *Provide solutions to no more than 6 questions of your choice.*
- *Use a different notebook for each question, clearly making each notebook with the corresponding question number.*

ENGLISH VERSION

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FORMULAS AND CONVERSION

$$\sum_{n=0}^{\infty} x^n \approx \frac{1}{1-x}, \quad |x| < 1$$

PHYSICAL CONSTANTS

$e = 1.602 \times 10^{-19}$ C
$m_e = 9{,}109 \times 10^{-31}$ Kg
$\hbar = 1.055 \times 10 \times 10^{-34}$ J·s
$k_B = 1.381 \times 10^{-23}$ J/K
$\varepsilon_0 = 8.854 \times 10^{-12}$ F/M
$\mu_0 = 4\pi \times 10^{-7}$ N/A^2

PHYSICS EQUATIONS :

$$\nabla \cdot \mathbf{D} = \rho_f$$

$$\nabla \cdot \mathbf{B} = 0$$

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$$

$$\nabla \times \mathbf{H} = \mathbf{J}_f + \frac{\partial \mathbf{D}}{\partial t}$$

$$\mathbf{B} = \frac{\mu_0}{4\pi} \int \frac{\mathbf{J} \times \hat{\mathbf{r}}}{|\mathbf{r}|^2} \, dV$$

$$f_{FD}(E) = \frac{1}{e^{(E-\mu)/k_B T} + 1}$$

$$f_{BE}(E) = \frac{1}{e^{(E-\mu)/k_B T} - 1}$$

Clausius-Mossotti relation

$$\frac{n\alpha}{3\varepsilon_0} = \frac{\varepsilon_r - 1}{\varepsilon_r + 2}$$

MATHEMATICS EQUATIONS

Integrals

$$\int_0^{\infty} \sqrt{x}\; e^{-x}\, dx = \frac{\sqrt{\pi}}{2}$$

$$\int_0^{\infty} e^{-ax^2}\, dx = \frac{1}{2} \sqrt{\frac{\pi}{a}}$$

$$\int_0^{\infty} x^2\, e^{-ax^2} dx = \frac{\sqrt{\pi}}{4a^{3/2}}$$

Law of cosines

$$c^2 = a^2 + b^2 - 2ab \cos \gamma$$

Stirling's approximation

$$n! \approx n^n e^{-n} \sqrt{2\pi n}$$

Identity

$$\nabla^2 \mathbf{A} = \nabla(\nabla \mathbf{A}) - \nabla \times \nabla \times \mathbf{A}$$

Trigonometric identities

$$\sin(\alpha \pm \beta) = \sin\alpha \cos\beta \pm \cos\alpha \sin\beta$$
$$\cos(\alpha \pm \beta) = \cos\alpha \cos\beta \mp \sin\alpha \sin\beta$$

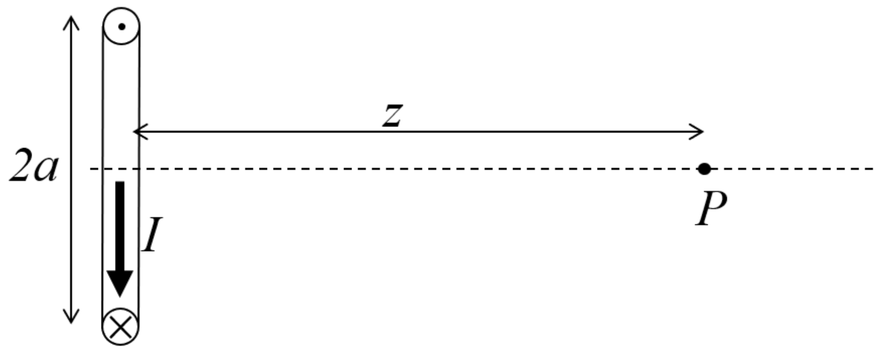
$$\sin 2\theta = 2 \sin\theta \cos\theta$$
$$\cos 2\theta = \cos^2\theta - \sin^2\theta$$

$$\cos\alpha \cos\beta = \frac{1}{2}\left(\cos(\alpha+\beta) + \cos(\alpha-\beta)\right)$$

$$\sin\alpha \sin\beta = \frac{1}{2}\left(\cos(\alpha-\beta) - \cos(\alpha+\beta)\right)$$

$$\sin\alpha \cos\beta = \frac{1}{2}\left(\sin(\alpha+\beta) + \sin(\alpha-\beta)\right)$$

QUESTION 1 : **ELECTROMAGNETISM**



Consider a conducting ring of radius a, carrying a constant current I, as sketched above.

a) **(5 points)** Write down the law which is used to compute the resultant magnetic field $\boldsymbol{B}$ generated by a contour of constant current I. Define all the relevant terms.

b) **(2 points)** Sketch the lines of the $\boldsymbol{B}$ field around the ring. Without performing calculations, provide argumentation for the direction of the $\boldsymbol{B}$ field on the symmetry axis of the ring, i.e. a dotted line in the drawing above.

c) **(6 points)** Apply the general expression in part (a) to find the $\boldsymbol{B}$ field at point P, which is situated on the symmetry axis of the ring at a distance z away from the center, as sketched above. Plot the z-dependence of the components of $\boldsymbol{B}(z)$.

d) **(3 points)** Now suppose that we form a solenoid - a tightly wound helix of conducting wire, where each wire turn, to an excellent approximation, can be considered to produce the $\boldsymbol{B}$ field of a conducting ring discussed in parts (b-c). Sketch the lines of $\boldsymbol{B}$ field around a solenoid of finite length L. Without performing calculations, provide an explanation and a qualitative sketch for the z-dependence of the components of $\boldsymbol{B}(z)$.

e) **(4 points)** Compute the $\boldsymbol{B}$ field of a solenoid for $L \to \infty$, considering n wire turns per unit length.

QUESTION 2 : QUANTUM MECHANICS

At $t = 0$, the wavefunction of a particle in the potential $V(x) = \frac{1}{2}m\omega^2 x^2$ is

$$\Psi(x, t = 0) = A \sum_n \left(\frac{1}{\sqrt{2}}\right)^n \psi_n(x),$$

where $\psi_n(x)$ are stationary eigenstates with eigenvalues $E_n = \hbar\omega\left(n + \frac{1}{2}\right)$ and $n = 0, 1, 2, \ldots$ Assume that these eigenstates are mutually orthonormal, $\langle\psi_n|\psi_m\rangle = \delta_{n,m}$.

a) **(5 points)** Find the normalization constant A.

b) **(5 points)** Give the expectation value for the energy.

c) **(5 points)** Write an expression for the time dependent $\Psi(x, t)$.

d) **(5 points)** Show that the density probability $\Psi(x, t)$ is a periodic function of time. Find the temporal period.

Useful information:

$$\sum_n \left(\frac{1}{2}\right)^n = 2$$

$$\sum_n n\left(\frac{1}{2}\right)^n = 2$$

QUESTION 3 : **STATISTICAL PHYSICS**

Figure 1 shows a model of a long molecular chain formed by n similar chemical structures. Under the effect of an external force F, the length parallel to the direction of the chain for each element is equal to ℓ_o. As shown in the figure, each element of the chain may have two non-degenerate energy states, i.e. a horizontal or vertical state with the corresponding energies of $-Fl_o$ and 0. By expressing the total chain length as $n \cdot x$. where $0 < x < \ell_o$ (see the figure), you need to:

a) **(4 points)** Derive the partition function for this system;

b) **(6 points)** Find the expression of the entropy of the chain as a function of x;

c) **(8 points)** Find the total average length of the chain as a function of temperature;

d) **(2 points)** Analyze your results and determine the conditions for which this molecular chain obeys the Hooke law found in the Polytechnique logo. "**Ut tensio sic vis**"(**Important note**: To answer this question you must assume that the elongation work given by the product Fl_o is much lower than k_BT).

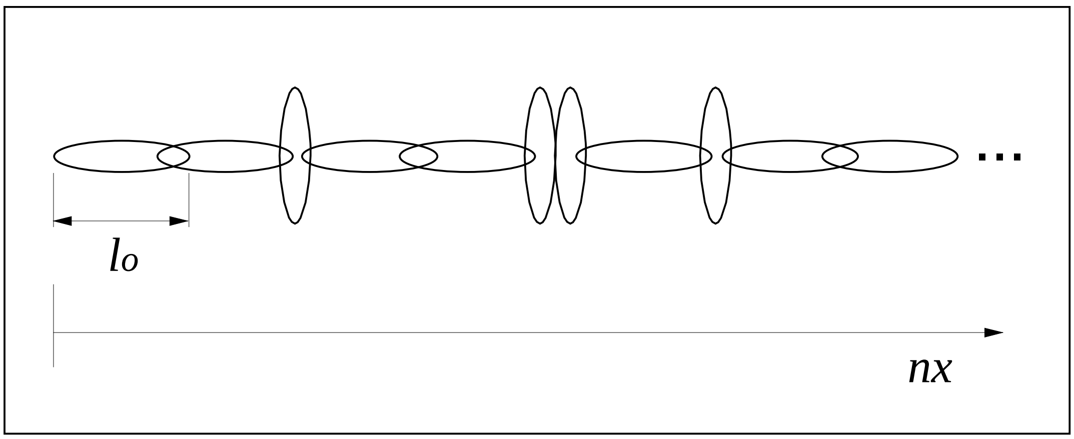
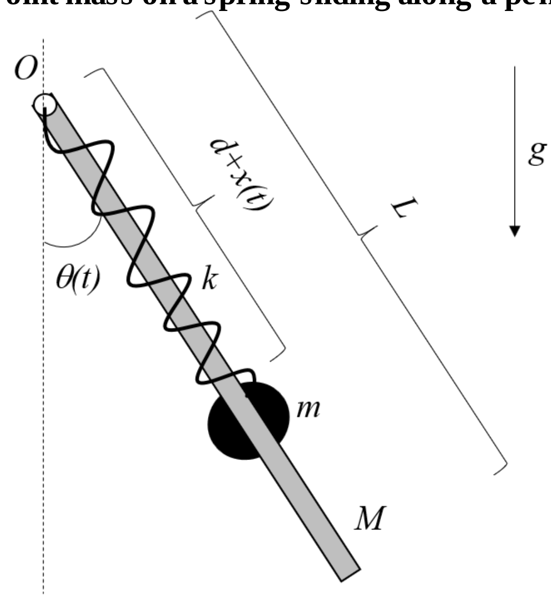


Figure 1. Model of a long molecular chain.

Note : To do your calculations, you can use the following approximation: $e^x \approx 1 + x$, with $x << 1$.

QUESTION 4 : **CLASSICAL MECHANICS**

Theoretical Mechanics. Point mass on a spring sliding along a pendulum.



Consider a pendulum in the form of a slender uniform cylinder of length L, mass M, and moment of inertia $I=ML^2/3$ (with respect to point O) that also features a massless spring with a spring constant k wound around it. One of the spring's ends is attached at the origin O, while the other one is attached to a point mass m that is free to slide along the rod without friction. The equilibrium length of a spring without point mass is d. Let the extended spring have length $d + x(t)$ and let the angle between the rod and the vertical be $\theta(t)$. Assume that the motion takes place in a vertical plane under normal gravity with the free fall acceleration given by g. Using the notions of Lagrange mechanics find the equations of motion for $x(t)$ and $\theta(t)$.

a) **(5 points)** Write expression for the total kinetic energy of a system.

b) **(5 points)** Write expression for the total potential energy of a system.

c) **(5 points)** Write expression for the Lagrangian and two corresponding equations of motion for $x(t)$ and $\theta(t)$ (Euler-Lagrange equations).

In the limit of small angles $\theta(t)<<1$ of pendulum oscillations and small displacements of a point mass, one can linearize the Euler-Lagrange equations and solve them exactly using:
$\theta(t)=A_\theta \exp(i\omega_\theta t+\phi_\theta)$ and $x(t)= mg/k+A_x \exp(i\omega_x t+ \phi_x)$

d) **(5 points)** Linearize equation of motion for $\theta(t)$ by assuming that $\sin(\theta(t)) = \theta(t)$ and consider that amplitude and velocity of the point mass oscillations are negligibly small $A_x \to 0$, $\partial x/\partial t \sim A_x \cdot \omega_x \to 0$. Find linearized equation of motion for $\theta(t)$, as well as expression for the frequency of oscillations of a pendulum ω_θ.

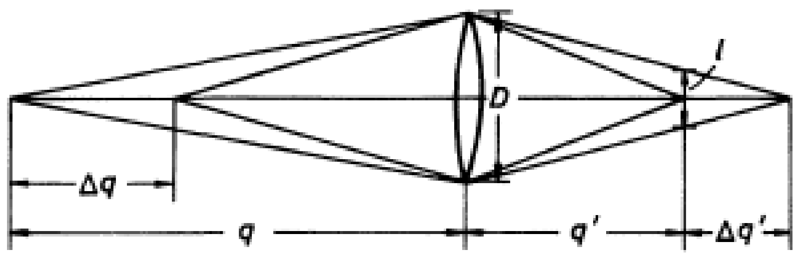
QUESTION 5 : **OPTICS I**

(20 points)

For a camera lens, the *depth of field* quantifies how far a point object (at a distance q) can be away from the position where it would be precisely in focus and still have the light from it fall on the film (at a distance q') within a *circle of confusion* of diameter l.

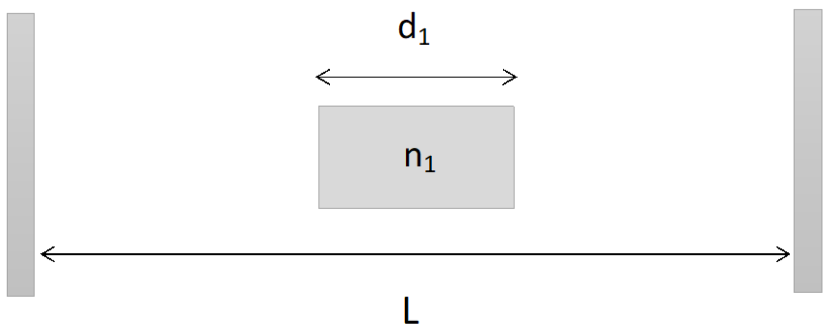
For a given image, derive a mathematical relation for the *depth of field* $\Delta q'$ as a function of the object distance q, the focal distance of the lens f, the f-number of the lens (defined as $N=f/D$, where D is the lens diameter), and the diameter of the circle of confusion.

Note: You can consider the object distance to be much larger than the focal length.



QUESTION 6 : **OPTICS 2**

Consider the optical cavity shown below. It is composed of 2 mirrors with (intensity) reflection coefficients R and a dielectric medium with refractive index n_1 and length d_1. The two mirrors are separated by a distance L.



a) **(5 points)** Calculate the free spectral range of the cavity.

b) **(5 points)** Calculate the lifetime of a photon in this cavity. The photon lifetime corresponds to the time required for the energy in the cavity to decrease by a factor $1/e$.

The following 2 questions are independent of your answers in a) and b).

Consider a situation where the dielectric medium is pumped. This causes the cavity to oscillate in 2 modes with frequencies ν_1 and ν_2. The corresponding electric fields are separately measured outside of the cavity using a detector equipped with a spectral filter. They are given by $E(t) = E_1 \cos(2\pi\nu_1 t + \varphi_1(t))$ and $E(t) = E_2 \cos(2\pi\nu_2 t + \varphi_2(t))$. The spectral filter is then removed.

c) **(5 points)** Find an expression for the intensity on the detector as a function of time if the phases φ_1 and φ_2 are constant in time.

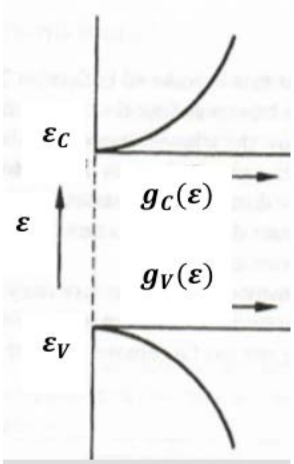
d) **(5 points)** Find an expression for the intensity on the detector if the phases φ_1 and φ_2 fluctuate randomly and independently. You can consider that your detector is slow with respect to the fluctuations.

QUESTION 7 : **SOLID STATE PHYSICS I**

Consider an intrinsic semiconductor. Let ε be the energy of an electron, $g_C(\varepsilon)$ the density of states in the conduction band and $g_V(\varepsilon)$ the density of states in the valence band (see the figure below). Assume $\varepsilon_C - \varepsilon_F \gg k_B T$ and $\varepsilon_F - \varepsilon_V \gg k_B T$ and

$$g_C(\varepsilon) = A_1(\varepsilon - \varepsilon_C)^{\frac{1}{2}}$$
$$g_V(\varepsilon) = A_2(\varepsilon_V - \varepsilon)^{\frac{1}{2}}$$

Where ε_C represents the energy of the bottom of the conduction band, ε_V the energy of the top of the valence band, ε_F the Fermi energy, T the temperature, k_B the Boltzmann constant, A_1 and A_2 two constant prefactors.



a) **(7 points)** Find an expression for the number of electrons in the conduction band (n) as a function of ε_C, ε_F, k_B, T and A_1.

b) **(7 points)** Find an expression for the number of holes in the valence band (p) as a function of ε_V, ε_F, k_B, T and A_2

c) **(6 points)** Write down an explicit expression for $\varepsilon_F(T)$.

Annex

$\int_0^\infty e^{-x} x^{1/2} dx = \frac{1}{2}\sqrt{\pi}$

QUESTION 8 : **SOLID STATE PHYSICS II**

Consider a classic model of a solid consisting of n independent atoms, each composed of one electron bound to an infinitely massive nucleus by a spring (obeying Hooke's law), with a natural frequency ω_0. Dissipative processeses are modeled by a friction force $\boldsymbol{F} = -\Gamma\boldsymbol{v}$, where $\boldsymbol{v}$ is the velocity of the electrons and $\Gamma > 0$. Consider response of this solid to a small external electromagnetic field of angular frequency ω.

a) **(4 points)** Write down a classical equation of motion for the displacement of electrons from the equilibrium as a function of the applied electromagnetic field. Define all the terms in the final expression. (Hint: provide justification for considering electronic response only to the electric component of the incident field).

b) **(6 points)** Derive the expression for the real and imaginary parts of the dielectric permittivity. Sketch the two quantities as a function of frequency and assign the physical significance to each.

c) **(3 points)** Write down the expression for the dielectric permittivity obtained in (b) in the limit of vanishing ω_0. Discuss the material class described by such response and identify characteristic frequency and a timescale with the familiar quantities from the Drude model.

d) **(4 points)** Determine the reflectivity R of the material described in (c) in the limit of $\Gamma \rightarrow 0$. Provide a sketch of $R(\omega)$.

e) **(3 points)** Based on the visual appearance of gold, provide an explanation for how one can estimate the value of the plasma frequency in this metal? How do you expect this value to change for silver?