A Finite Element Method for Moving-Habitat Models in 2D Space

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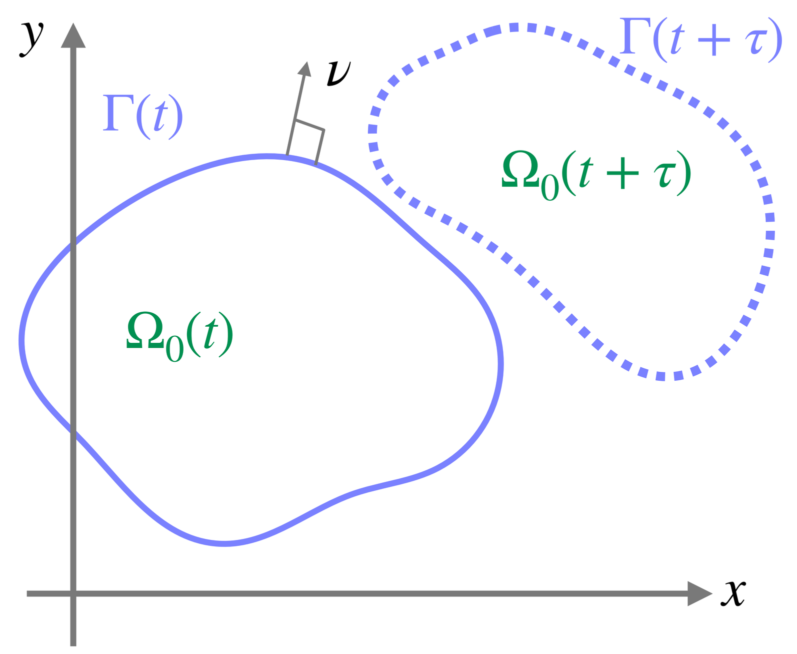
MOVING-HABITAT MODELS



- Mathematical models that help us understand the conditions under which a species persists in the face of climate change
- A bounded domain represents the suitable habitat, outside this domain represents the unsuitable habitat
- The boundary of the suitable habitat moves in time to represent the moving temperature isoclines

$$\partial_t u_0 = D_0 \Delta u_0 + u_0(r - a u_0), \qquad (x,y) \in \Omega_0(t),$$

$$\partial_t u_1 = D_0 \Delta u_1 - m u_1, \qquad (x,y) \in \Omega_1(t) = \mathbb{R}^2 \backslash \bar{\Omega}_0,$$

$$u_0 = \kappa u_1 \qquad (x,y) \in \Gamma(t),$$

$$D_0 \partial_\nu u_0 + u_0 c \cdot \nu = D_1 \partial_\nu u_1 + u_1 c \cdot \nu, \quad (x,y) \in \Gamma(t)$$

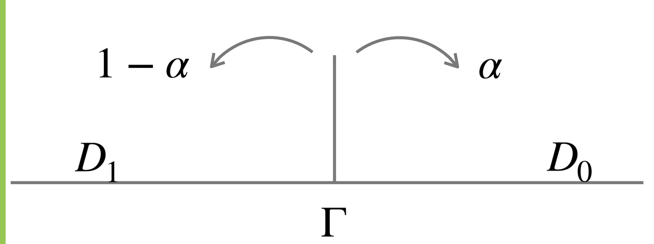
$$\lim_{|(x,y)| \to \infty} u(x,y,t) = 0.$$

THE JUMP AND FLUX CONDITION

Density jump: $u_0 = \kappa u_1$

$$\kappa = \frac{\alpha}{1-\alpha}\sqrt{\frac{D_1}{D_0}},$$

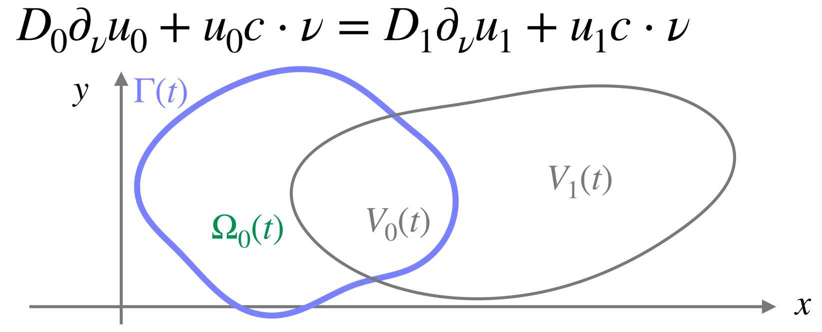
α = probability that an individual moves into the suitable habitat when at the habitat edge.



In 1D space, derivable via a random walk [Maciel&Lutscher2013AmNat]

Flux condition:

$$D_0 \partial_\nu u_0 + u_0 c \cdot \nu = D_1 \partial_\nu u_1 + u_1 c \cdot \nu$$



Change of mass in V:

$$\frac{d}{dt}\int_V u \mathrm{d}(x,y) = -\int_{\partial V} F(u) \cdot n \mathrm{d}S + \int_V G(u) \mathrm{d}(x,y)$$

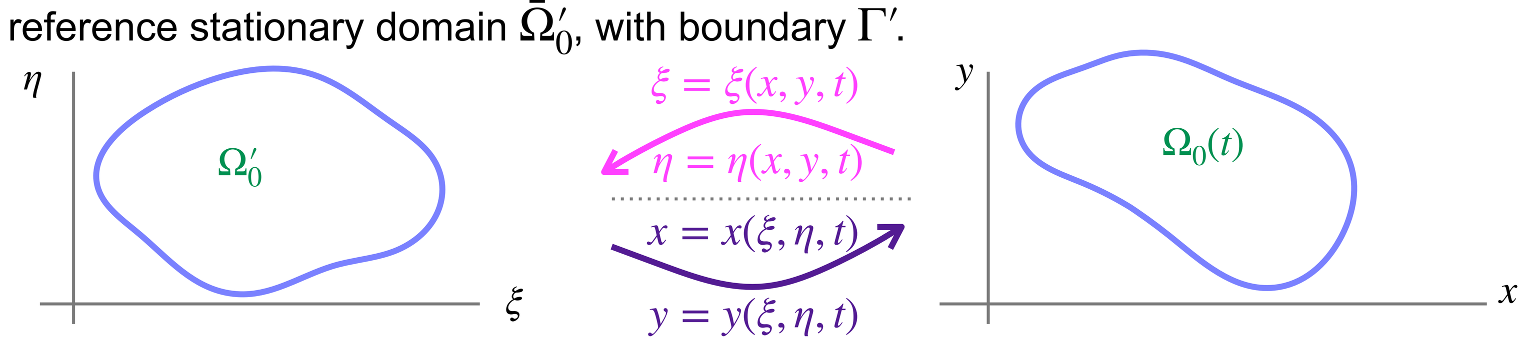
Fick's law for diffusive flux: $F(u) = -D(x,y)\nabla u$

Reynold's transport theorem:

$$\frac{d}{dt}\int_{V(t)} u \mathrm{d}(x,y) = \int_{\partial V(t)} u v \cdot n \mathrm{d}S + \int_{V(t)} \partial_t u \mathrm{d}(x,y)$$

REFERENCE FRAME

We assume that for each $t \in [0,T]$, that the moving domain, $\bar{\Omega}_0(t)$, is the image of a reference stationary domain $\bar{\Omega}_0'$, with boundary Γ'.



$$\partial_t w = D_i \left[w_{\xi\xi}\left((\xi_x)^2 + (\xi_y)^2\right) + w_{\eta\eta}\left((\eta_x)^2 + (\eta_y)^2\right) + w_\xi \Delta\xi + w_\eta \Delta\eta + 2 w_{\xi\eta}\left(\xi_x \eta_x + \xi_y \eta_y\right)\right]$$
$$- \left((\xi_t, \eta_t) \cdot \nabla(\xi,\eta)\right) w + G(w), \qquad (\xi,\eta) \in \mathbb{R}^2 \backslash \Gamma',$$

$$D_0 (\nu \cdot \nabla) w_0 + w_0 c \cdot \nu = D_1 (\nu \cdot \nabla) w_1 + w_1 c \cdot \nu, \qquad (\xi,\eta) \in \Gamma',$$

$$(\nu \cdot \nabla) w_i = w_{i\xi}\left(\nu_1 \xi_{x_1} + \nu_2 \xi_{x_2}\right) + w_{i\eta}\left(\nu_1 \eta_{x_1} + \nu_2 \eta_{x_2}\right),$$

$$D_i = \begin{cases} D_0, & (\xi,\eta) \in \Omega_0' \\ D_1, & (\xi,\eta) \in \Omega_1' = \mathbb{R}^2 \backslash \bar{\Omega}_0', \end{cases} \qquad G(w) = \begin{cases} w(r - aw), & (\xi,\eta) \in \Omega_0' \\ -mw, & (\xi,\eta) \in \Omega_1' = \mathbb{R}^2 \backslash \bar{\Omega}_0' \end{cases}$$

OBJECTIVES

- Write an efficient finite element scheme to accurately capture the jump in density.
- Employ the finite element scheme to study the transient and asymptotic dynamics of the moving-habitat model

METHODOLOGY

We consider the case where Ω_0 moves only by linear translations $x - c_1 t$ and $y - c_2 t$. We discretise time. Then in the reference frame we propose a **finite element method** for the system:

$$\frac{w_i^{n+1} - w_i^n}{\tau} = d_i \Delta w^{n+1} + (c \cdot \nabla) w^{n+1} + G(w^n), \qquad (\xi,\eta) \in \mathbb{R}^2 \backslash \Gamma',$$
$$w_0^{n+1} = \kappa(x,y) w_1^{n+1}, \qquad (\xi,\eta) \in \Gamma', \qquad (1)$$
$$d_0 \partial_\nu w_0^{n+1} + (c \cdot \nu) w_0^{n+1} = d_1 \partial_\nu w_1^{n+1} + (c \cdot \nu) w_1^{n+1}, \qquad (\xi,\eta) \in \Gamma',$$
$$w_1^{n+1} \to 0, \text{ as } |(\xi,\eta)| \to \infty.$$

Function spaces:

- $X = \{v \in L^2(\Omega_0' \cup \Omega_1') \mid v|_{\Omega_i'} = v_i \in H^1_*(\Omega_i')\} \simeq \Pi_{i=0}^1 H^1_*(\Omega_i');$ $H^1_*(\Omega_0') = H^1(\Omega_0')$, and $H^1_*(\Omega_1') = H^1_0(\Omega_1')$.
- $V_\kappa = \{v \in X : v_0 = \kappa v_1$ on Γ and $v = 0$ on $\partial\Omega_1 \backslash \Gamma\}$.
- $M = \{\mu \in \Pi_{i=0}^1 H^{-1/2}(\Gamma) : \exists q \in H(\text{div}; \Omega)$ such that $q \cdot \nu_i = \mu$ on $\Gamma\}$.

Classical variational formulation:

Find $w_i^{n+1} \in V_\kappa$ such that

$$\sum_{i=0}^1 \int_{\Omega_i} d_i \nabla w_i^{n+1} \nabla v_i + \left(\left(c \cdot \nabla v_i\right) + \frac{1}{\tau_n} v_i\right) w_i^{n+1} \mathrm{d}(\xi,\eta) = \sum_{i=0}^1 \int_{\Omega_i} \left(G(w_i^n) + \frac{w_i^n}{\tau_n}\right) v_i \mathrm{d}(\xi,\eta),$$

for all $(v_1, v_2) \in V_1$.

Hybrid formulation:

Find $(w^{n+1}, \lambda) \in X \times M$ such that

$$a(w^{n+1}, v) + b_1(v, \lambda^{n+1}) = \mathcal{G}(w^n, v), \qquad \forall v \in X,$$
$$b_\kappa(w^{n+1}, \mu) = 0, \qquad \forall \mu \in M.$$

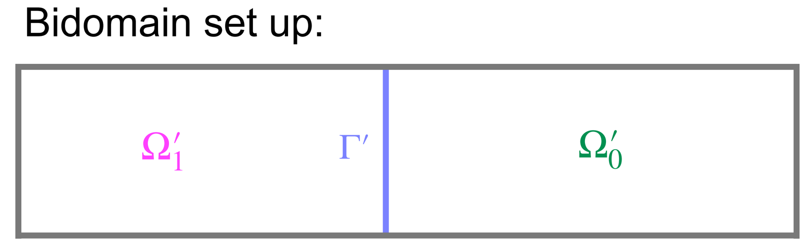
Here, $a(w,v) = \sum_{i=0}^1 \int_{\Omega_i} d_i \nabla w_i \nabla v_i + \left(\left(c \cdot \nabla v_i\right) + \frac{1}{\tau_n} v_i\right) w_i \mathrm{d}(\xi,\eta),$

$\mathcal{G}(w,v) = \sum_{i=0}^1 \int_{\Omega_i} \left(G(w_i) + \frac{w_i}{\tau_n}\right) v_i \mathrm{d}(\xi,\eta)$, and $b_\kappa(w,\lambda) = < \lambda, w_0 - \kappa w_1 >_{H^{-1/2}(\Gamma), H^{1/2}(\Gamma)}$.

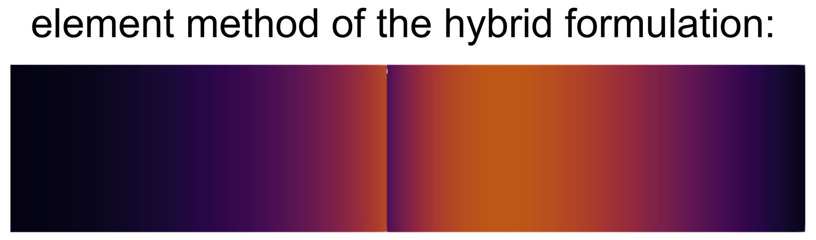
VALIDATION

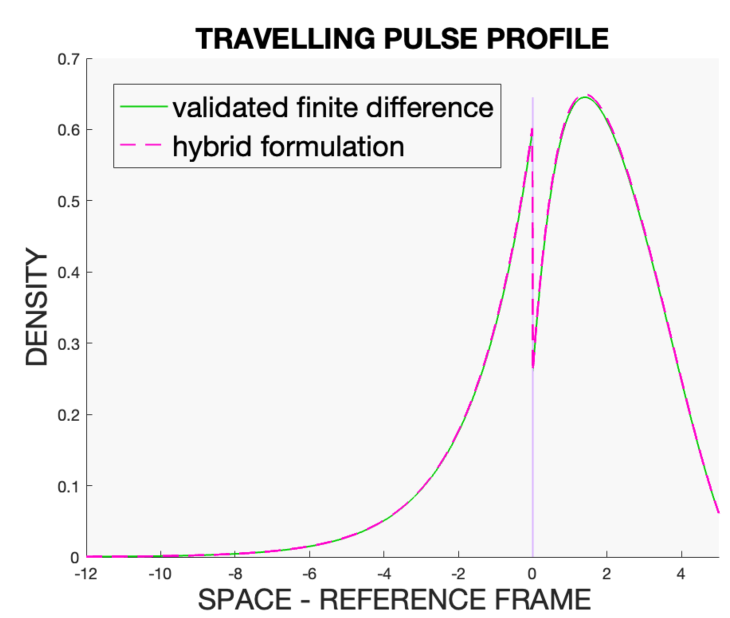
Using FreeFEM we validate the **hybrid formulation** by comparing the solutions with the ones from the 1D space finite difference scheme constructed and validated in our previous work [MacDonald et al, 2021, Math Biosci].

Bidomain set up:



Travelling pulse solution given by the finite element method of the hybrid formulation:





Existence and Uniqueness

Lemma:
A continuous linear functional L on the space X vanishes on V_κ if and only if there exists a unique element $\mu \in M$ such that

$$\forall v \in X,\ L(v) = \int_\Gamma \mu(v_0 - \kappa v_1) \mathrm{d}S.$$

Theorem:
The hybrid problem has a unique solution $(w, \lambda) \in X \times M$. Moreover, $w \in V_\kappa$ is the solution of system (1) and

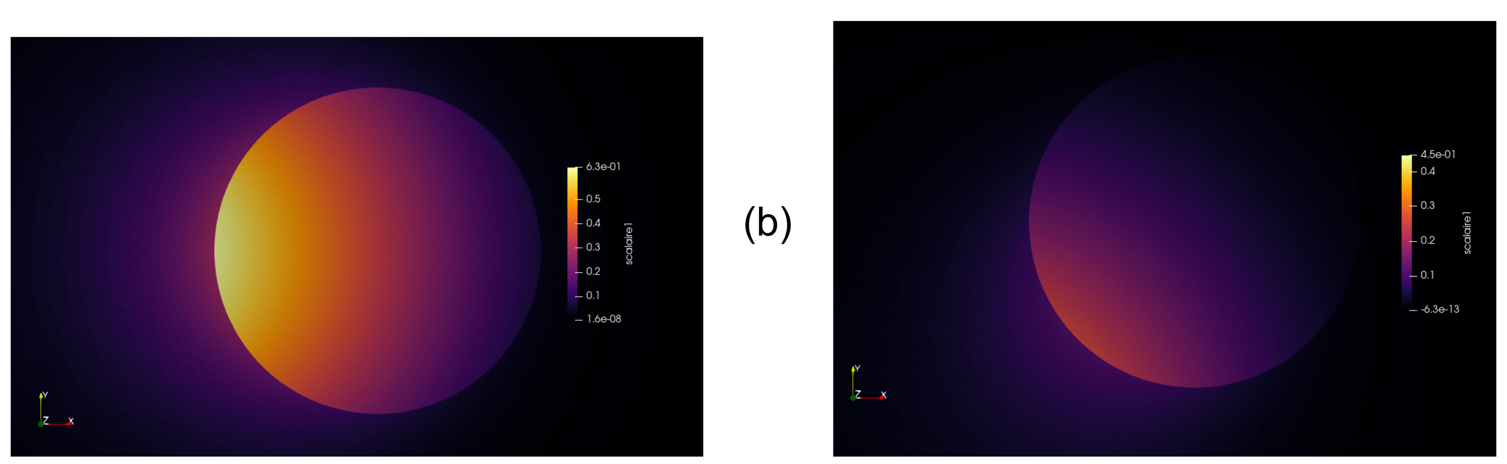
$$\lambda = \frac{\partial w_i}{\partial \nu_i} \text{ on } \Gamma \text{ for } i = 0,1.$$

Proof:
Done for $c = 0$. We need to extend for $c \neq 0$. The proof relies on the lemma and follows the ideas in Raviart and Thomas, 1977.

First simulations

Simulations results with the hybrid finite element scheme. Model parameters are the same across both figures, except that the shifting speed of Γ is larger in Figure (b) than that of Figure (a). At the fast shifting speed, we see much smaller local densities.

(a) (b)



NEXT STEPS

- Take $c \neq 0$ for the existence and uniqueness statements
- Prove: Via the Banach-Nečas-Babuška Theorem, prove that the classical variational formulation is well-posed.
- Error analysis of the finite element scheme for the hybrid system
- Simulations!

NOMENCLATURE

$u(x,y,t)$: density of population in space and time
Ω_0: suitable habitat
Ω_1: unsuitable habitat
Γ: habitat edge (interface between habitat types)
$D_i > 0$: diffusion coefficient in Ω_i, $i = 0, 1$
$r > 0$: intrinsic growth rate
$a > 0$: intraspecific competition coefficient
$m > 0$: mortality rate
c: velocity of interface
$\partial \nu$: unit normal derivative

REFERENCES AND ACKNOWLEDGEMENTS

References for moving-habitat models in 1D space: **MacDonald et al, 2021, Math Biosci**; **MacDonald and Lutscher, 2018, J Math Biol**; Potapov and Lewis, 2004, Bull Math Biol; Berestycki et al, 2008, Bull Math Biol.
References for hybrid formulations: Raviart and Thomas, 1977, Math Compute; Nicolaides, 1982, SIAM J Numer Anal; Belgacem, 1999, Numer Math. Ern and Guermond, Theory and Practise of Finite Elements, Springer, 2000