ABSTRACT

This paper presents the synthesis of optimally unstable two-phalanx underactuated fingers. The method to obtain a design unable to grasp almost any object under normal conditions is presented. The technique relies on the careful analysis of the grasp-state plane and equilibrium curve associated to two-phalanx underactuated fingers. First, results of the analysis of the grasp-state plane are recalled. Second, unstable configurations and paradoxical equilibrium are presented. Then, the synthesis of optimally unstable fingers is detailed. Finally, applications of these unstable fingers are proposed.

INTRODUCTION

The lack of success of complex robotic hands in the past has mainly been attributed to the cost and complexity of these systems with numerous actuators and sensors requiring a cumbersome control architecture. In order to overcome these limitations, a particular emphasis has been placed on the reduction of the number of degrees of freedom, thereby decreasing the number of actuators. In the past few years, many prototypes have been proposed taking advantage of underactuation applied to grasping. The latter involves a smaller number of actuators without decreasing the number of degrees of freedom and can be implemented through the use of passive elements like springs and a dedicated transmission technique. The main advantage of this method is that it leads to a mechanical adaptation of the finger to the shape of the object grasped [1,2,3,4,5,6]. This adaptation is realized without a dedicated control law, e.g. a constant actuator torque is usually sufficient to achieve the shape-adaptation.

Underactuation in Grasping

Underactuation in robotic fingers is different from the concept of underactuation usually presented in robotic systems and both notions should not be confused. An underactuated robot is generally defined as a manipulator with fewer actuators than degrees of freedom. On the other hand, underactuated fingers generally use elastic elements in their “unactuated” joints. Thus, one should rather think of these joints as uncontrollable or passively driven instead of unactuated. In an underactuated finger, the actuation wrench $T_a$ is applied to the input of the finger and is transmitted to the phalanges through suitable mechanical components, e.g. four-bar linkages, pulleys and tendons, gears, etc. Since underactuated fingers have many degrees of freedom, and fewer actuators, passive elements are used to kinematically constrain the finger and ensure the shape-adaptation of the finger to the object grasped. To this end, springs and mechanical limits are often used. An example of underactuated two-phalanx finger using linkages and its closing sequence are illustrated in Fig. 1. The actuation torque $T_a$ is applied to the first link which transmits the effort to all phalanges. Notice the mechanical limit that allows a pre-loading of the spring to prevent any undesirable motion of the second phalanx and also to prevent hyperextension of the finger. Springs are useful for keeping the finger from incoher-
ent motion, but when the grasp sequence is complete, they still oppose the actuation. Thus, springs shall be designed with the smallest stiffness possible, however sufficient to keep the finger from collapsing. In practical prototypes, one has to ensure that grasps are stable in the sense that ejection is prevented. Indeed, an ideal grasping sequence as illustrated in Fig. 1 does not always occur. For in the final configuration some phalanx forces may be negative. If one phalanx force is negative the corresponding phalanx will lose contact with the object. Then, another step in the grasping process will take place: the remaining phalanges corresponding to positive forces will slide on the object surface. This sliding process will continue until either a stable configuration is achieved, or the finger will curl away and loose contact with the object (ejection, Fig. 2). The authors have proposed in previous papers a method to analyze this motion and predict if a stable grasp will be achieved or if ejection will occur [7, 8]. However, this theory has been considered difficult to understand and with little practical utility. It is the opinion of the authors that this not true and this paper intends to prove it. After a brief introduction of the grasp stability theory—a more detailed presentation can be found in [9]—, some peculiar degeneracy in the stability conditions are presented here for the first time. These degeneracies are thereupon used to synthesis optimally unstable fingers. Although, obtaining unstable fingers may seem useless, applications are presented in the last part of the paper.

Figure 1. Closing sequence of a two-phalanx finger with linkage transmission.

Figure 2. Example of an ejection sequence.

Introduction to the Grasp Stability Theory

First and foremost, the definition of a stable grasp should be established. In this paper as well as in the previous works of the authors [7, 8], a stable grasp means that the finger is in static equilibrium. More precisely, this situation happens if the finger is in contact with an object and the contact forces generated by the finger are positive. Furthermore, if there is no contact with a certain set of phalanges, the corresponding contact forces generated by the finger should be zero in order for the finger to be in static equilibrium. Contrary to the usual concept of “stability” used in grasping analysis, a stable grasp will hereinafter designate a grasp situation where all phalanx forces are positive or zero. Note that it is assumed that the object is fixed in space and therefore the stability property applies to the finger and only to the finger. A model for the fingers under study is presented in Fig. 3. To distribute the single axis actuation to the two phalanges, several transmission mechanisms have been proposed in the past: linkage [2], tendon and pulleys [1,10], or more recently, cam-tendon [11], as illustrated in Fig. 3.

Each of these transmission mechanisms can be characterized by a Transmission matrix \( T \), as defined in [12], i.e.,

\[
T = \begin{bmatrix} 1 & R \\ 0 & 1 \end{bmatrix}
\]  

where \( R \) is the transmission ratio characteristic of the transmission technique used [12]. The latter is a function of the design parameters and possibly, of angle \( \theta_2 \). For instance,

\[
\begin{cases} 
R = -h/(h+l_1) & \text{with a four-bar linkage} \\
R = -r_2/r_1 & \text{with pulleys/cams}
\end{cases}
\]
where \( h \) is the directed distance between point \( O_1 \) and the geometric intersection of lines \((OO_1)\) and \((P_1P_2)\) (this distance, illustrated in Fig. 4, can be negative). In the case of the cam-tendon finger, the tendon-driven expression remains valid but both pulley radii are complex functions of the angles \( \theta_1 \) and \( \theta_2 \). However, the proximal cam is usually a simple pulley, eliminating the influence of \( \theta_1 \) [11]. This simplification does not reduce the range of possible values for \( R \) since the latter is defined as the ratio \(-r_2/r_1\). For instance, in [11], the distal cam radius is chosen such that

\[
R = -\frac{1}{4\cos^2 \left( \frac{\theta_2}{2} \right)}
\]  

which yields \( f_1 = f_2 \) for every angle \( \theta_2 \) (if the contacts are assumed to be located at mid-phalanx), a situation defined as force-isotropy [8]. The Transmission matrix allows the computation of the contact forces generated by the finger very easily, namely one has [12]

\[
f = J^{-T}g + t
\]

where \( f^T = [f_1, f_2]^T \) is the vector of the contact force expressions, \( t^T = [T_a, T_2]^T \) is the input torque vector exerted by the actuator \( (T_a) \) and the spring between the phalanges \( (T_2) \). Matrix \( J \) is the Jacobian matrix of the grasp [12], i.e.,

\[
J = \begin{bmatrix}
k_1 & 0 \\
k_2 + l_1(\cos \theta_2 + \mu \sin \theta_2) & k_2
\end{bmatrix}
\]

where \( \mu \) relates the tangential contact force \( f_2 \) to the normal contact force \( f_2 \), i.e., \( f_2 = \mu f_2 \). Variables \( k_1 \) and \( k_2 \) characterize the locations of the contact points and are defined in Fig. 3. This coefficient is usually unknown, but has on the edge of the friction cone (and only on the edge of the friction cone) a particular value, namely the static friction coefficient \( \mu = \pm \mu_{\text{static}} \). From Eqn (4), one obtains

\[
f = \begin{bmatrix}
k_2(1+\mu) + Rl_1 G T_a - k_2 + l_1 \mu T_2 \\
-\frac{R}{k_2} T_a + \frac{k_1}{k_2} T_2
\end{bmatrix},
\]

where \( G = (\cos \theta_2 + \mu \sin \theta_2) \). With the expressions of the contact forces, one can study the conditions under which both \( f_1 \) and \( f_2 \) are positive (or not), which depends on the contact situation, namely the pair \((k_2, \theta_2)\), but neither on \( \theta_1 \) (if no spring are located in the base joint \( O \)) nor on \( k_1 \). This pair defines a plane, where the contact situation can be tracked. This plane is referred to as the grasp-state plane and \((k_2, \theta_2)\) as the grasp-state [9].

When in an unstable configuration (one of the phalanx forces is negative), the closing process will force the finger to lose contact with the phalanx with a negative force, usually the proximal one. With tendon-driven fingers, the distal force is always positive while linkage-driven fingers can have a negative distal force but only in rare (and usually outside of the workspace) configurations [9]. If the proximal contact force is negative, contact will only remain with the distal phalanx, and the finger will slide against the object. An equilibrium position can still be attained but just for a one and unique particular position of contact \( k_2 \). This is, of course, the position corresponding to the solution of the equation \( f_1(k_2) = 0 \), i.e.,

\[
k_2 = e = -\frac{R}{1+\mu} l_1(\cos \theta_2 + \mu \sin \theta_2)
\]

if the spring is neglected. For instance, with tendon-driven fingers, one obtains

\[
e = \frac{r_2}{r_1 - r_2} l_1(\cos \theta_2 + \mu \sin \theta_2).
\]

**Contact trajectories and Equilibrium Curves**

If the proximal contact is lost, a sliding motion is undergone by the distal phalanx of the finger in contact with the object and one has to determine if a stable position will be reached or not. To obtain this evolution, the contact location as seen from the distal phalanx will be introduced. It can be easily shown by considering the triangle constituted by \( O_1, O_2 \) and the contact point (illustrated in Fig. 5), that if only this contact exists and if it is
fixed in space, one has [9]

\[ k_2^2 - k_{2i}^2 + 2l_1 (k_2 \cos \theta_2 - k_{2i} \cos \theta_{2i}) = 0 \]  \hspace{1cm} (9)

where \( k_{2i} \) and \( \theta_{2i} \) define an arbitrary initial configuration, for example, the precise instant when the contact on the first phalanx is lost. This equation expresses that the distance between the base point of the finger and the contact location is constant for any pair \((k_2, \theta_2)\).

Eqn. (9) allows to obtain the evolution of the contact position with respect to the evolution of \( \theta_2 \) during the sliding motion. In these situations, the finger now has one degree of freedom while keeping contact with the object, this motion has been previously noticed and referred to as self-posture changing motion [13]. The contact situation evolves on the contact trajectory defined by Eqn. (9) and also by its relative location with respect to the equilibrium position, which determines the sign of \( f_1 \). If the contact trajectory crosses an equilibrium curve, the grasp is finally stable, or else contact with the object will be lost, namely one obtains the ejection phenomenon due to the kinematic evolution \((k_2 \geq k_2 \), illustrated in Fig. 2). The equilibrium curves are either defined by Eqn. (7) or by mechanical limits on \( \theta_2 \). Indeed, mechanical limits are equilibrium curves: if the finger is resting against a mechanical limit, the latter is in static equilibrium since no further motion is possible. The finger in such a position acts like a simple rotational gripper with an unusual shape of the jaw. However, the shape adaptation in these situations is not optimal. In the grasp-state plane, mechanical limits on \( \theta_2 \) are represented by horizontal lines. Practically, mechanical limits are of the utmost importance, almost noticeably to prevent opening-ejection [8]. Depending on the geometric parameters of the mechanism, one can draw the final stability of the grasp depending on the initial contact situation, an example is presented in Fig. 6.

For instance, if the initial contact is established in configuration \( A \) (indicated in Fig. 6), the two-phalanx contact configuration is unstable (the proximal force is negative). Then, contact with the proximal phalanx is lost and the finger undergoes a sliding motion described by the associated contact trajectory until, in this case, configuration \( B \) where the finger is in static equilibrium is reached. On the other hand, if the initial contact is in configuration \( C \), once again the proximal force is negative and its associated contact lost, but in this case, the sliding motion continues until configuration \( D \) is reached where the distal contact is also lost and the finger ejects the object. Finally, the stability regions can be constructed by inspecting if an arbitrary initial grasp-state will lead to a stable situation (equilibrium curve or mechanical limit) or not.

If friction is also considered during the grasp, Eqn. (7) is used to obtain the equilibrium locus [9]. However, there is no longer a single equilibrium curve but rather two curves. The sign of \( \mu \) defines these two curves in the grasp-state plane, each corresponding to one sliding direction, i.e., \( \mu = +\mu_{\text{static}} \) or \( \mu = -\mu_{\text{static}} \). The area between these two curves corresponds to a value of the coefficient \( \mu \) such that \(-\mu_{\text{static}} < \mu < +\mu_{\text{static}}\), hence to a static equilibrium.

**Unstable Designs and Paradoxical Equilibrium**

As discussed in the previous Section, the equilibrium curve is of utmost importance with underactuated finger as it defines
a locus of possible static equilibrium even in case of missing contact. Associated with contact trajectories defined by the shape and position of the object to grasp, the equilibrium curve characterizes the capability of underactuated fingers to generate stable grasps. If this curve is poorly designed, no equilibrium will be attained during a grasp and the ejection of object to be grasped will occur which renders the finger useless. Hence, the analysis of the equilibrium equation defined with Eqn (7) is crucial when designing underactuated fingers. Another important aspect, and previously unaddressed issue, to consider about the equilibrium equation is its possible degeneracy, namely when:

\[ e(\theta_2) = \pm \infty. \]  

(10)

The latter can be either local, i.e., for a particular \( \theta_2 \) or architectural, for all \( \theta_2 \). Eqn. (7) which expresses the equilibrium equation for any kind of transmission of our model, relates the degeneracy condition to the transmission coefficient, i.e.,

\[ e(\theta_2) = \pm \infty \iff R(\theta_2) = -1. \]  

(11)

For instance, considering a tendon-driven finger, if \( a = 1 \), then \( e = \infty \). What is important to notice is that in this case, the degeneracy is not local, but global (for all \( \theta_2 \))! For linkage-driven fingers, the same phenomenon happens if \( c = a \). Such designs have been referred to as evil twins in [12] because of their penchant to ejection. However, and contrary to what was previously assumed [7], these designs do have equilibrium positions! Namely

\[
\text{if } R = -1, \ f_1 = 0 \text{ if and only if } G(\theta_2) = 0.
\]

Therefore, if friction is neglected, the equilibrium curve degenerates into a set of two straight lines defined by \( \theta_2 = \pm \pi / 2 \). Hence, these positions correspond to a peculiar equilibrium where any contact location leads to a static equilibrium (illustrated in Fig. 7)! Nevertheless, the price to pay in return is that for these fingers, only two configurations can lead to equilibrium, i.e., the shape adaptation is not achieved. If the degeneracy was only local, i.e., \( R(\theta_2) = -1 \) for a particular angle \( \theta_2 \), in the latter configuration (locally achievable with a linkage-driven finger), no equilibrium is possible, except if \( G(\theta_2) = 0 \) for the same angle, which is not likely (yet possible) to happen. Therefore, the equilibrium location is indeed pushed to infinity. Hence, the configurations close to a degeneracy location should be generally avoided since, in their neighbourhood, the equilibrium locus corresponds to large values of \( k_2 \), not physically achievable. These situations are therefore prone to closing-ejection [9].

If \( R > 1 \), the equilibrium curve is negative if \( -\pi/2 < \theta_2 < \pi/2 \) and positive elsewhere, therefore no equilibrium position is physically on the phalanx for the common small values of \( \theta_2 \). This situation arises when \( r_2 > r_1 \) with tendon-driven fingers and when \( c > a \) with linkage-driven fingers. However, it should be noted that these designs are not always unstable: for example, if \( \theta_2 > \pi/2 \) a design with \( \alpha > 1 \) can lead to a stable grasp. Nevertheless such designs are generally avoided since they do not have equilibrium positions in the usual range of motion of the finger. For this reason they are the basic elements of the synthesis of optimally unstable fingers proposed in this paper.

![Figure 7. Unstable design in a paradoxical equilibrium with (right) and without (left) friction.](image)

**Synthesis of an Optimally Unstable Finger**

**Unstable grasp conditions, sign of the contact force**

The first condition to synthesize an unstable gripper is to ensure that no initial grasp-state will generate a positive proximal contact, i.e. \( f_1 < 0 \ \forall (k_2, \theta_2) \). This condition can be developed using Eqn. (6), namely

\[ k_2 (1 + R) + Rl_1 G < 0 \]  

(12)

by taking into account only the part of \( f_1 \) that can change sign. It is assumed that \( -\pi < \theta_2 < \pi \) and \( 0 < k_2 < l_2 \). Hence, one obtains a condition on the transmission factor \( R \) to ensure the negativity of \( f_1 \), i.e.

\[ R < \frac{-k_2}{k_2 + l_1 G} = \frac{-1}{1 + \frac{\mu}{k_2}} \]  

(13)

since \( k_2 + l_1 G > 0 \) assuming—initially—that \( \mu = 0 \). Considering Eqn. (13), it can be seen that the condition obtained on \( R \) cannot be satisfied for any value of the grasp-state unless \( R = -\infty \) for grasp-states defined by \( (k_2, \arccos(-k_2/l_1)) \). However, no transmission ratio is known to be function of the distal contact...
location \( k_2 \), hence the condition
\[
R \left[ \arccos \left( \frac{k_2}{I_1} \right) \right] = -\infty \quad (14)
\]
cannot be satisfied. Thus, one must restrict the range of values of \( \theta_2 \) inside the workspace defined by \( -\pi/2 < \theta_2 < \pi/2 \) which yields that both \( I_1 G \) and \( k_2 \) are positive. Therefore, a sufficient condition on \( R \) to satisfy Eqn. (13) is
\[
R(\theta_2) < -1 \quad \forall \theta_2 \quad (15)
\]
This condition is easily achievable in practice using \( r_1 < r_2 \) with tendon-driven fingers or \( a < c \) with linkage-driven fingers. Thus, one could assume that an unstable design has been obtained. Indeed, if one considers the limit case \( c = a \) with a linkage-driven finger only in contact with the object through the distal phalanx, the latter is subjected to three pure forces, two of them being parallel. The condition to equilibrium, namely that the three forces intersect in a common point, seems impossible. Nevertheless, as discussed before this finger does have equilibrium configuration for \( \theta_2 = \pm \pi/2 \), i.e., at the limits of the workspace. Physically, these configurations correspond to the three forces intersecting at infinity. Furthermore, these limit values of \( \theta_2 \), which are equilibrium curves, cannot be dismissed as marginal because of the contact trajectories. Indeed, many contact trajectories corresponding to an initial contact location in the finger workspace will lead to \( \theta_2 = \pi/2 \) as illustrated in Fig. 8. Except for contact situations in the shaded area, any other initial grasp-state (between the contact asymptotes) will lead to a static equilibrium with \( \theta_2 = \pi/2 \).

Hence, considering only the sign of the contact forces is fundamentally incomplete and might be misleading in the analysis of the stability of underactuated fingers.

**Back to the grasp-state plane**

In order to synthesize unstable fingers, instead of considering only the sign of the proximal contact force, one has to carefully analyze the grasp-state plane. To obtain an unstable design, one has to ensure that for any initial grasp-state, the sliding motion on the associated contact trajectory will not converge to an equilibrium location. For this condition to be valid in any typical initial condition \((-\pi/2 < \theta_2 < \pi/2 \) and \( 0 < k_2 < l_2 \)), no contact trajectory must intersect an equilibrium curve. Graphically, no equilibrium curve should appear in the grasp-state plane between the two asymptotes of the contact trajectories [9] as illustrated in Fig. 8. The latter intersect the equilibrium curve defined by \( f_1 = 0 \) in two points defined by \( \theta_2 = \pm \pi/2 \) and \( k_2 = 0 \) according to Eqn. (7) (except when friction is considered). Hence, in order for the equilibrium curve not to enter the central part of the grasp-state plane, two conditions are necessary:

1. the equilibrium location for \(-\pi/2 < \theta_2 < \pi/2 \) should be negative,
2. the slope of the equilibrium curve for \( \theta_2 = \pm \pi/2 \) should be greater than the slope of the respective trajectory asymptote.

The first condition can be easily related to the transmission ratio \( R \) using Eqn. (7), namely
\[
e \left[ \frac{-\pi}{2}, \frac{\pi}{2} \right] < 0 \Leftrightarrow R \left( \left[ \frac{-\pi}{2}, \frac{\pi}{2} \right] \right) \notin [-1, 0]. \quad (16)
\]

It should be noted that in order to create a closing motion of the distal phalanx, \( R \) has to be negative according to Eqn. (6). Hence, the previous condition can be reduced to \( R < -1 \), the condition obtained with the first method. The second and additional condition can be developed as
\[
\frac{\partial e}{\partial \theta_2} \bigg|_{\theta_2=\pi/2} > \frac{\partial g}{\partial \theta_2} \bigg|_{\theta_2=\pi/2} \quad (17)
\]
where the asymptotes are defined by \( g = k_2 = -2l_1 \cos \theta_2 \). Eqn. (17) can be developed as
\[
R(\pi/2) \frac{l_1}{1 + R(\pi/2)} > 2l_1. \quad (18)
\]
It should be noted that only \( \theta_2 = \pi/2 \) has been considered since the contact trajectories and asymptotes are symmetrical with respect to the axis \( \theta_2 = 0 \). Hence, Eqn. (18) is valid for both asymptotes...
totes. Finally, one obtains

\[ R(\pi/2) < -2. \]  

(19)

Hence, a sufficient condition for the non-existence of an achievable equilibrium location between the contact trajectory asymptotes is \( R(\theta_2) < -2 \) in this area of the grasp-state plane. The condition obtained in the previous Section \( (R < -1) \) is incomplete as it only guarantees that the proximal contact force will be negative in the chosen workspace but the edges of this workspace are stable curves leading to a static equilibrium. On the other hand, if the conditions obtained with the analysis of the grasp-state plane are followed, the proximal contact force will be negative and ejection will occur for any grasp-state in this workspace. Similarly, mechanical limits should be chosen in order not to be located between the asymptotes, as they also constitute equilibrium curves, i.e.,

\[ \theta_{2\text{max}} > \arccos \left( \frac{-l_2}{2l_1} \right). \]  

(20)

Fig. 9 shows different equilibrium curves with a constant transmission ratio \( R \) (tendon-driven fingers), the critical value of \( R = -2 \) is clearly illustrated (all curves are symmetrical with respect to the axis \( \theta_2 = 0 \)).

Fig. 10. Stable design (left) and unstable prototype (right) of two-phalanx underactuated finger.

Friction can be considered in the analysis, but its influence is predictably stabilizing on the grasp. An example of stability regions taking into account a friction coefficient of 0.5 is presented in Fig. 11.

Figure 9. Equilibrium curves for different values of \( R \).

Figure 11. Stability regions due to friction \( (R = -5, \mu = \pm 0.5) \).
Fig. 11. Note that the grasp-state corresponding to a low value of $k_2$ will eventually reach the equilibrium curve indicated by $\mu = -0.5$.

Applications

Finally, potential applications of unstable grippers are discussed in this section. Unstable grippers are not usually particularly desirable unless as a toy. Indeed, one could imagine building plastic robotic hands such as the one illustrated in Fig. 12 to actually amuse children and/or trick people.

Another application to unstable grasping—albeit morally condemnable—is confidence games, e.g. skill crane machines. An example of such a machine is illustrated in Fig. 13. They are common in fairs and commercial malls. For a small fee, the player can control a crane equipped with a gripper to pick a gift, usually a plush or a small toy, and has to drop it in a place where he/she can grab it. If such machines were equipped with a gripper designed using the analysis described in the previous section, the user’s chance to actually obtain a gift is almost zero. Note that the objects to grasp are a statistically predictable data, for instance, in Fig. 13, only balls with a certain diameter and friction coefficient are to be considered. This a-priori data can be used to refine the design obtained in the previous section, for instance, contact trajectories corresponding to a circular object can be taken into account [9].

Conclusion

In this paper, the synthesis of an optimally unstable two-phalanx underactuated finger has been presented. The method to obtain a design unable to grasp almost any object under normal conditions has been presented. Two different approaches have been considered to this aim, the first method only considers the sign of the contact forces while the second is based on the analysis of the grasp-state plane. It has been shown that the first method is incomplete and might be misleading. Hence, the usefulness of the grasp-state plane has been illustrated to obtain underactuated finger designs. Potential applications of the unstable designs obtained in this paper were proposed as toys and/or hoax skill crane machines. This paper illustrates the concepts developed in [9] and emphasizes the importance of the grasp-state analysis. New material presented in this paper consists of the identification of the possible degeneracies of the equilibrium equation and their application to the synthesis of unstable grippers. Finally, a word of warning: the authors do not recommend to use the technique presented in this paper to actually trick people out of their money. Prospective swindlers should meditate this advice from a distributor of skill crane machines:

“You can have the best plush toys in the world in your equipment, but if your customers do not win often enough, the game will not be profitable. [...] If you tighten your game up too much and no one wins, then everyone will find out and quit playing.”
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