Abstract—In this paper, a methodology is proposed for the analysis of the force capabilities of connected differential mechanisms. These systems are the key elements used to extend the principle of underactuation in grasping from the fingers to the hand itself. The concept of underactuation in robotic grasping—\textit{with} fewer actuators than degrees of freedom (DOF)—allows the hand to adjust itself to an irregularly shaped object without complex control strategies and sensors. Several technological solutions have been proposed in the past but no theoretical background has been provided to analyze their characteristics, especially with respect to the forces generated. The purpose of this paper is to provide such a theoretical foundation and to illustrate its usefulness with examples applied to grasping. First, several differential elements are presented and studied. Second, a mathematical method to obtain the output force capabilities of connected differential mechanisms is presented. Finally, the technique presented is applied to two types of underactuated robotic hands.

I. INTRODUCTION

Common robotic hands do not usually consist of one single finger, except maybe in the case of tentacle inspired systems. Most prototypes have a number of fingers comprised between two and five. The purpose of the underactuation between the fingers is to use the power of one actuator to drive the open/close motion of all the fingers of a robotic hand collectively. The transmission mechanism used to achieve such a property must be adaptive, i.e., when one or more fingers are blocked, the remaining finger(s) should continue to move. When all the fingers are blocked, the force should be well distributed among the fingers and it should be possible to apply large grasping forces while maintaining a stable grasp. Introducing underactuation between the fingers of a robotic hand allows to reduce the complexity of the systems, from the actuation point of view. The application of this principle has been demonstrated with several prototypes [1]–[6]. The basic element commonly used to this end is the differential mechanism. According to the IFToMM terminology [7], a differential mechanism is a two-DOF mechanism that may resolve a single input into two outputs and vice versa. In underactuated grasping systems in the sense used in this paper, a spring element is generally used to constrain kinematically the outputs of the differential mechanism in its pre-grasping phase. Usually, the spring is of negligible stiffness with respect to the actuation torque and used to keep both outputs in the same kinematic state (e.g., same angles or same positions). However, this is not obligatory, especially if multiple outputs are provided through stacked elements. It should also be noted that differential mechanisms, if the most common element used in underactuation, are not the only technological solution to achieve this property in grasping [8].

In this paper, several differential elements specifically used in robotic hands to provide underactuation between a certain number of fingers are presented. In the first part of the paper, the focus is placed on the analysis of several common differential mechanisms modeled as basic force input/output cells (illustrated in Fig. 1). These solutions are studied with respect to their force capabilities: the objectives and methods of this analysis are similar to those of [9] where phalanx forces were studied. Indeed, if underactuation can be used to drive several robotic fingers, it can also be used to drive several phalanges of these fingers with a single actuator. Although similar in its approach, the force analysis of underactuation between the fingers is fundamentally different from the underactuation in the fingers as will be shown. In the second part of the paper, some mathematical tools are presented to analyze the force capabilities of connected differential mechanisms, used to provide \( n \) outputs by stacking multiple elements. These tools also include a simple method to study the implications of the reversal principle of differential mechanisms. Finally, examples are thereupon provided to clearly illustrate the results of the paper and their application to the design of grasping devices. To the best of the authors’ knowledge the force capabilities developed by connected differential mechanisms have never been mathematically studied, despite having been used in numerous occasions. The idea itself of connecting differential mechanisms to produce multiple output adaptive systems is however not new and should be attributed to Prof. Hirose in [10], [11].

\[ F_o \]

\[ F_{1} \]

\[ F_{2} \]

\[ \text{transmission mechanism} \]

Fig. 1. Basic force transmission differential cell.
II. Design Solutions

A. Movable pulley

The movable pulley is perhaps the most well-known and commonly used mechanical element to distribute one actuation force to two outputs [10], [12]. Since a tendon is used, such a system can easily be employed to drive robotic fingers which commonly use tendons for actuation transmission. In Fig. 2, for instance, two n-phalanx underactuated fingers are driven with one input through a movable pulley located inside the palm of the hand. Note that the fingers are themselves also underactuated since several phalanges are driven with a single tendon. The movable pulley can also drive fully-actuated fingers (using coupled rotations for example). More generally, this principle can be used to drive any mechanical system driven by two tendons and thereby provide adaptability. The notation pertaining to the analysis of the movable pulley presented in this section is shown in Fig. 3. It is pointed out that the two DOFs of the pulley are: an horizontal translation and a rotation about an axis perpendicular to the plane of the figure. The input force is noted \( \text{F}_a \) while the two output forces are respectively \( \text{F}_{1a} \) and \( \text{F}_{2a} \). A spring, modeled by the torque \( T_s \), constrains the remaining DOF of the pulley. The purpose of this analysis is to obtain the actuation forces transmitted to the output as functions of the input forces—the spring is considered an input—, i.e.

\[
\text{F} = \text{T}^f_t \text{t}^s
\]

with

\[
\text{F} = \begin{bmatrix} \text{F}_{1a} \\ \text{F}_{2a} \end{bmatrix} \quad \text{t}^s = \begin{bmatrix} \text{F}_a \\ T_s \end{bmatrix}.
\]

Matrix \( \text{T}^f_t \)—hereafter referred to as the force transmission matrix—characterizes the transmission device used. Referring to Fig. 3 and using a very simple static analysis, one readily obtains

\[
\text{T}^f_t = \frac{1}{c} \begin{bmatrix} r & \sin \alpha_2 \\ r & -\sin \alpha_1 \end{bmatrix}
\]

where \( c \) is the sum of the respective distances from points \( A_1 \) and \( A_2 \) to the axis of the prismatic joint, i.e.

\[
c = r(\sin \alpha_2 + \sin \alpha_1).
\]

\[
F_1 = F_2 = \frac{F_a}{\sin \alpha_1 + \sin \alpha_2}
\]

An important property of the movable pulley is that it is force-isotropic, in the sense of the discussion of [13], i.e., the two output forces are equal. This result was expected since the tension in the common cable is constant. Furthermore, the system is globally force-isotropic since this property holds for any configuration, i.e. any value of \( \alpha_1 \) and \( \alpha_2 \). The force transmission ratio—sometimes refer to as the mechanical advantage of the system—\( F_i^a/F_a \) for \( i = 1, 2 \) is however significantly affected by the angles of the output tendons even over a limited range of the angular values, as illustrated in Fig. 4.

B. Seesaw mechanism

The principle of this device is to use a central seesaw bar whose translation in one direction is controlled while the other is prevented by design. The rotation of the bar is used to accommodate the difference of position between two output rods transmitting the motion. The notation pertaining to this mechanism is indicated in Fig. 5. Seesaw mechanisms have been successfully used in a number of underactuated hands. It is referred to as the “differential lever” in [1] or the “equalizing bar” in [4]. Several different names have been used in the literature to refer to the mechanism analyzed in this section. In this paper, the denomination proposed in [10] is used, namely,
obtains by expressing the conservation of pressure in the fluid. One force transmission matrix of a T-pipe can be readily written of a fluid to separate its flow into two distinct streams. The IFToMM definition, and takes advantages of the deformability outputs may be to use a T-pipe scheme, as illustrated in Fig. 6.

C. Fluidic T-pipe

The simplest method to distribute one input towards two outputs may be to use a T-pipe scheme, as illustrated in Fig. 6. This fluidic stage is a differential mechanism, according to the IFToMM definition, and takes advantages of the deformability of a fluid to separate its flow into two distinct streams. The force transmission matrix of a T-pipe can be readily written by expressing the conservation of pressure in the fluid. One obtains

$$\begin{bmatrix} F_1^a \\ F_2^a \end{bmatrix} = S_a \begin{bmatrix} S_1^a/S_a \\ S_2^a/S_a \end{bmatrix} \begin{bmatrix} F_1^o \\ F_2^o \end{bmatrix}$$

(8)

where $S_a$, $S_1^a$ and $S_2^a$ are the respective section areas of the input, primary and secondary output pipes. It is noted that in this case, instead of having a matrix characterizing the underactuation capability of the device, a vector is rather obtained. This is due to the fact that no return torque or force is usually embedded in the stage itself. Namely, the system is not fully constrained from a static point of view. However, in practical applications, a return spring is commonly found in the fluidic struts driven by this stage. When this system is used to drive robotic fingers, it is usually desired to keep both

output states identical. In the latter case, if a spring opposes the difference between the two outputs, one obtains

$$\begin{bmatrix} F_1^a \\ F_2^a \end{bmatrix} = \begin{bmatrix} S_1^a/S_a & -1/S_a \\ S_2^a/S_a & 0 \end{bmatrix} \begin{bmatrix} F_1^o \\ F_2^o \end{bmatrix}$$

(9)

where the force transmission matrix is readily recognizable. Alternatively, if two springs act on the outputs, the previous equation becomes

$$\begin{bmatrix} F_1^a \\ F_2^a \end{bmatrix} = \begin{bmatrix} S_1^a/S_a & 0 & 0 \\ S_2^a/S_a & 0 & 1 \end{bmatrix} \begin{bmatrix} F_1^o \\ F_2^o \end{bmatrix}$$

(10)

Both cases are illustrated in Fig. 7. In the case where two springs are used, the force transmission matrix of the stage is not square, due to the fact that the device is overconstrained with respect to a static analysis. Indeed, a $n$-output $m$-input underactuated mechanism in the sense of this paper requires $n - m$ springs in order to be statically determined. However, in the latter case, the two-output one-input device uses not one but two springs. This overconstraint may seem penalizing since it implies that for a desired output in force, an infinite number of solutions of the input force vector are possible. However, this is usually not true since the stiffness of the springs is fixed by design and therefore the only control variable available to the user is the input force. Namely, the force inputs of the springs are usually not controllable. Another characteristic of the T-pipe scheme is that the output forces are constant and independent of the output position (if the spring(s) is (are) negligible) with respect to the actuator force. Furthermore, the force-isotropic property can be easily achieved with this device by making the output section areas equal.
### D. Planetary and bevel gear differentials

Commonly found differential mechanisms are based on either planetary or bevel gear transmissions. Aside from automotive systems, gear differentials have also been used in underactuated hands. For instance, bevel gear differentials have been used in [14] to drive a copy of the SARAH hand [6], while the latter uses planetary gear trains. In the planetary differential, the input is arbitrarily chosen to be the carrier torque while the primary and secondary outputs are the sun and ring torque, as illustrated in Fig. 8. Therefore, one obtains

\[
\begin{bmatrix}
T_1^a \\
T_2^a
\end{bmatrix} = T^f \begin{bmatrix}
T_s \\
T_r
\end{bmatrix}
\]

(11)

with the force transmission matrix

\[
T^f = \begin{bmatrix}
\frac{r_c}{r_a + r_c} & 1 \\
\frac{r_s}{r_a + r_c} & -1
\end{bmatrix}
\]

(12)

where \(r_c\) and \(r_s\) are respectively the ring/annulus and sun pitch radii. Note that the authors use the term “force” transmission matrix even if only torques are considered. The number of teeth can be equivalently used in the above equation since both quantities are related through the module of the gear train

\[
2i = mN_i, \quad \text{where } N_i \text{ is the number of teeth of gear } i.
\]

By inspection of the planetary gear force transmission matrix it is clear that the output torques are constant and independent of the output position, similarly to the T-pipe scheme presented in Section II-C. Again, similarities between technological solutions stand out from the analysis. The independence of the output force from the output kinematic state can be of interest to ensure constant performances over the driven device’s workspace. The bevel-gear differential is simply a particular case of the more general planetary gear differential. In this special case, both output torques are globally force-isotropic and are equal to one half of the input torque (if the spring is neglected). It is noted that force isotropy is impossible to achieve with a simple planetary gear differential because it would require a zero radius of the planet gear, according to eq. (12). However, particular planetary gears, where the planet consists of two rigidly connected gears instead of a single gear, have been used to overcome this problem. An example of such gear trains achieving force-isotropic outputs is used in the SARAH class prototypes [6]. The latter use one-input three-output devices based on two stacked planetary gear differentials, as illustrated in Fig. 9, to drive three underactuated fingers with one actuator.

### III. Combining multiple stages

#### A. Transmission tree analysis

In order to obtain \(n\) outputs—typically three to four in robotic hands—one has to stack multiple differential devices, each stage adding one degree of freedom to the system. Any layout tree may be used for the routing of the actuation, as illustrated in Fig. 10. The structure of the transmission tree used in underactuated fingers (and not between them) is often strictly serial. This simplification results from the physical layout of robotic fingers where rigid links (the phalanges) are connected to each other in series. Indeed, the serial architecture of the fingers naturally leads to a serial underactuated transmission. Nevertheless, this serial architecture is not necessary and is almost certainly not used if the driven system differs from a mechanical finger. Furthermore, the mechanical connection between the phalanges leads to peculiar phenomena [9], [15], such as the vanishing of certain phalanx forces leading to uncontrolled motion of the finger. This uncontrolled motion can, under certain conditions, degenerate into a sequence where the finger ejects the object instead of seizing it.

To produce \(n\) outputs, \(n - 1\) differential stages are required since each differential mechanism produces two outputs for one input. Each transmission stage can be described by its associated force transmission matrix, defined in eq. (1), hence one obtains \(n - 1\) equations. Using a superscript \(i\) to indicate
In this Section, all the transmission matrices are assumed to be square. If this hypothesis cannot be satisfied—c.f. the discussion about the fluidic stage (Section II-C)—the right-hand side vector should be modified accordingly. Non-square matrices do not prohibit the use of the method presented in this Section since no matrix inversion is required. The form of the single input-output equation depends on the transmission tree layout. For example, if the latter is strictly serial, as illustrated in Fig. 11(a), one obtains

$$\mathbf{F} = \mathbf{T}_i \mathbf{t} \quad i = 1, \ldots, n-1.$$  

(13)

Therefore, one obtains \(n-1\) equations that can be combined in a single input-output relationship, namely

$$\begin{bmatrix}
F_1^n \\
F_2^n \\
\vdots \\
F_n^n
\end{bmatrix} = \mathbf{T}_i \begin{bmatrix}
F_1 \\
T_1 \\
\vdots \\
T_{n-1}
\end{bmatrix}$$

(14)

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$$\mathbf{F}_i = \prod_{i=1}^{n-1} \begin{bmatrix}
1_{n-1-i} \\
0
\end{bmatrix} \mathbf{T}_i \begin{bmatrix}
0 \\
1_{i-1}
\end{bmatrix}$$

(15)

The matrices included in the above product are block diagonal, i.e., square diagonal matrices in which the diagonal elements are square matrices, and the off-diagonal elements are zero. The blocks on the diagonal are either the force transmission matrix of the associated stage or the identity matrix. Indeed, \(1_k\) indicates the identity matrix of dimension \(k\). Note that, if one has \(1_0\), no component exists. The proof of eq. (15) is not included in this paper due to the lack of space and can be found in [16]. On the other hand, if the tree is strictly symmetrical, as illustrated in Fig. 11(b), one has

$$\mathbf{T}_i = \prod_{i=1}^{2^i-1} \begin{bmatrix}
2^{i-1} \mathbf{T}_f \\
\vdots \\
0
\end{bmatrix} \begin{bmatrix}
0 \\
1_{n-2^i}
\end{bmatrix}$$

(16)

With this transmission tree, \(n\) is obligatory a power of two, i.e., \(n = 2^k \, k \in \mathbb{N}\). The proof of eq. (16) is also not given in this paper due to the lack of space and can be found in [16]. Matrix \(i^k \mathbf{B}_k\) is the “bubble” matrix of rank \(i\) and dimension \(k\). Its purpose is to re-order the terms resulting from the multiplication of the block diagonal matrix with the vector on the right hand side of the equation, i.e., to move upwards the components of the vector that are not in their final position (hence the name “bubble”). The elements of this matrix are defined as

$$i^{p_{m,j}} = 1 \quad \text{if} \quad \begin{cases} m = 2k \\ k = 1, \ldots, 2^{i-1} \end{cases}$$

or

$$i^{p_{m,j}} = 1 \quad \text{if} \quad \begin{cases} m = 2k - 1 \\ k = 1, \ldots, 2^{i-1} \end{cases}$$

or

$$i^{p_{m,j}} = 0 \quad \text{otherwise}.$$  

(17)

It should be noted that \(i^k \mathbf{B}_k = 1_k\), i.e. the identity matrix of dimension \(k\). The bubble matrices are tedious to express properly but are not complicated once their function is clearly understood. Furthermore, matrices of higher dimensions can be written recursively, e.g.,

$$\mathbf{B}_4 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

(18)

$$\mathbf{B}_8 = \begin{bmatrix} \mathbf{B}_4 & \mathbf{0}_4 \\ \mathbf{0}_4 & \mathbf{1}_4 \end{bmatrix}$$

(19)

$$\mathbf{B}_{16} = \begin{bmatrix} \mathbf{B}_8 & \mathbf{0}_8 \\ \mathbf{0}_8 & \mathbf{1}_8 \end{bmatrix}$$

(20)
Hence, the global transmission matrix is

\[
\begin{bmatrix}
T_{1,1}^f & T_{1,2}^f & 0 \\
T_{2,1}^f & T_{2,2}^f & T_{2,2}^f \\
T_{3,1}^f & T_{3,2}^f & T_{3,2}^f \\
\end{bmatrix}
\]

(23)

where \(T_{i,j}^f\) is the component on the \(i^{th}\) line and \(j^{th}\) column of matrix \(T^f\). If the effect of the springs can be neglected, i.e., \(T_{i}^s = T_{2}^s = 0\), the latter equation becomes

\[
\begin{bmatrix}
F_a^0 \\
F_a^2 \\
F_a^3 \\
F_a^4 \\
\end{bmatrix} = \begin{bmatrix}
T_{1,1}^f & T_{1,2}^f \\
T_{2,1}^f & T_{2,2}^f \\
T_{3,1}^f & T_{3,2}^f \\
\end{bmatrix}
\]

(24)

With four outputs, the tree layout can be either serial or symmetrical. If the former case is chosen, one has

\[
\begin{bmatrix}
F_a^0 \\
F_a^2 \\
F_a^3 \\
F_a^4 \\
\end{bmatrix} = \begin{bmatrix}
1 & 0 & 0 & 1 \\
0 & 1 & 0 & 0 \\
0 & 0 & 3T^f & 0 \\
0 & 0 & 1 & 1 \\
\end{bmatrix}
\]

(25)

and thus, the global transmission matrix is

\[
\begin{bmatrix}
T_{1,1}^f & T_{1,2}^f & 0 & 0 \\
T_{2,1}^f & T_{2,2}^f & T_{2,2}^f & 0 \\
T_{3,1}^f & T_{3,2}^f & T_{3,2}^f & 0 \\
T_{3,1}^f & T_{3,2}^f & T_{3,2}^f & 0 \\
\end{bmatrix}
\]

(26)

If the effect of the springs is negligible, eq. (25) becomes

\[
\begin{bmatrix}
F_a^0 \\
F_a^2 \\
F_a^3 \\
F_a^4 \\
\end{bmatrix} = \begin{bmatrix}
T_{1,1}^f & T_{1,2}^f \\
T_{2,1}^f & T_{2,2}^f \\
T_{3,1}^f & T_{3,2}^f \\
\end{bmatrix}
\]

(27)

If a symmetrical layout is selected, one obtains

\[
\begin{bmatrix}
F_a^0 \\
F_a^2 \\
F_a^3 \\
F_a^4 \\
\end{bmatrix} = \begin{bmatrix}
T_{1,1}^f & 0 \\
0 & 3T^f \\
0 & 1 & 0 \\
0 & 0 & 1 \\
\end{bmatrix}
\]

(28)

and the global transmission matrix is

\[
\begin{bmatrix}
2T_{1,1}^f & T_{1,1}^f & T_{1,2}^f & T_{1,2}^f \\
2T_{2,1}^f & T_{2,1}^f & T_{2,2}^f & T_{2,2}^f \\
3T_{1,1}^f & T_{1,2}^f & T_{1,2}^f & T_{1,2}^f \\
3T_{2,1}^f & T_{2,2}^f & T_{2,2}^f & T_{2,2}^f \\
\end{bmatrix}
\]

(29)

In this case, if the effect of the springs is negligible, eq. (28) becomes

\[
\begin{bmatrix}
F_a^0 \\
F_a^2 \\
F_a^3 \\
F_a^4 \\
\end{bmatrix} = \begin{bmatrix}
2T_{1,1}^f & T_{1,1}^f & T_{1,2}^f \\
2T_{2,1}^f & T_{2,1}^f & T_{2,2}^f \\
3T_{1,1}^f & T_{1,2}^f & T_{1,2}^f \\
3T_{2,1}^f & T_{2,2}^f & T_{2,2}^f \\
\end{bmatrix}
\]

(30)

If the transmission tree is neither serial nor symmetrical, the analysis should be done considering the particular layout of the tree and using eq. (13).

IV. EXCHANGING INPUTS AND OUTPUTS

In the previous analyses, certain input/outputs of the mechanisms were arbitrarily chosen. While, for the movable pulley and the seesaw mechanism, one particular branch of the system stands out and was naturally identified as the input of the system, other transmissions such as the T-pipe or the gear differential do not present such a particularity. Indeed, our choice of input and outputs was purely empirical and even for the devices where one branch was immediately identified with the input, nothing prevents the designer from using the latter as an output. Therefore, the behaviour obtained when the input and outputs are exchanged should be analyzed for the sake of completeness. This analysis is rather simple since the basic equations describing the static equilibrium—i.e. the input/output relations—still hold with any set of input/outputs. Therefore, the equations described using the force transmission matrix still hold and only a re-arranging of the terms is required to obtain a new force transmission matrix. Assume a transmission stage characterized by its force transmission matrix \(T^f\). If the force output \(F_i^a\) is now the input, the new transmission matrix is

\[
\begin{bmatrix}
F_a \\
F_a^2 \\
F_a^3 \\
F_a^4 \\
\end{bmatrix} = \begin{bmatrix}
1 & \frac{1}{T_{1,1}^f} & -T_{1,2}^f \\
T_{2,1}^f & 1 & \frac{1}{T_{2,2}^f} \\
T_{3,1}^f & T_{3,2}^f & 1 \\
\end{bmatrix}
\]

(31)

And conversely, if \(F_i^a\) is now the input, one obtains

\[
\begin{bmatrix}
F_a^0 \\
F_a^2 \\
F_a^3 \\
F_a^4 \\
\end{bmatrix} = \begin{bmatrix}
1 & \frac{1}{T_{1,1}^f} & -T_{1,2}^f \\
T_{2,1}^f & 1 & \frac{1}{T_{2,2}^f} \\
T_{3,1}^f & T_{3,2}^f & 1 \\
\end{bmatrix}
\]

(32)

V. APPLICATIONS

A. Underactuated gripper

The design considered in this section is illustrated in Figs. 12 and 13. Similar underactuated grippers have been presented in [15] and [17]. Two underactuated linkage-driven fingers consisting of two phalanges are actuated through a seesaw mechanism as presented in Section II-B.

1) Force-isotropic design: seesaw mechanisms usually provide different values of force outputs, i.e., no force-isotropy except possibly in isolated configurations. However, this characteristic can be of use to obtain force-isotropic grasps in non-symmetrical situations. Indeed, if the typical grasped object is not symmetrical and/or the contact points are not symmetrically located with respect to axis of the gripper, the contact forces will not be equal if the actuation torque induced at the base of each finger is identical. Hence, to generate a
force-isotropic grasp where all contact forces are equal, the
outputs of the differential stage should be non force-isotropic.
This seemingly surprising statement arises from the fact that
the grasp configurations are not symmetric. The first step to
achieve a force-isotropic design is identical to the procedure
presented in [13]. Indeed, the force-isotropic condition, namely
\[ f_1 = f_2 = f_3 = f_4 \]  
(33)
includes the force-isotropic condition in each finger, i.e., \( f_1 = f_2 \) and \( f_3 = f_4 \). The details of the method having already been
presented, it is assumed here that the geometric parameters
of both fingers are known to satisfy a force-isotropic contact
configuration defined by
\[ (k_1, k_2, k_3, k_4, \theta_1, \theta_2, \theta_3, \theta_4) \]
(34)
and the contact forces \( (f_1, f_2, f_3, f_4) \) are therefore defined as
functions of the actuation torque at the base of each finger.
However, if this contact situation is not symmetrical, i.e.
\( k_1 \neq k_3, k_2 \neq k_4, \theta_1 \neq \theta_3, \) or \( \theta_2 \neq \theta_4 \), it implies that
\( f_1 \neq f_3 \) and \( f_2 \neq f_4 \) if the actuation torque transmitted
to the fingers is identical. To ensure that the final contact
configuration is actually force-isotropic, the output force of
the differential stage should satisfy
\[ \frac{d_2 F_2^a}{d_1 F_1^a} = \frac{f_1}{f_3} \]
(35)
where \( d_1 F_1^a \) and \( d_2 F_2^a \) are the torques generated by the
output forces \( F_1^a \) and \( F_2^a \) with respect to points \( O_1 \) and \( O_2 \)
respectively (see Fig. 13). Thus, \( d_1 \) and \( d_2 \) are respectively
the distances from line \( M \) to point \( O_1 \) and from line \( N \) to
point \( O_2 \), i.e., the output force lines of action with respect to
the associated finger. Hence, with the definition of the force
transmission matrix of the seesaw mechanism, eq. (6), the
differential stage has to satisfy
\[ \frac{d_2 b_1}{d_1 b_2} = \frac{d_2 a_1 \sin(\alpha_1 - \alpha)}{d_1 a_2 \sin(\alpha_2 - \alpha)} = \frac{f_1}{f_3} \]
(36)
to generate a force-isotropic grasp (the spring in the differential
stage is neglected in this section), with
\[ d_i = c_i \sin(\alpha_i - \theta_i^c) \quad i = 1, 2. \]
(37)
Therefore one can choose a set of geometric parameters for the
seesaw mechanism—including the location of the attachment
of the output rods to the base of the fingers \( c_1 \) and \( c_2 \)—that
satisfies the previous relationship, and obtain a force-isotropic
grasp. For instance, let us suppose that the desired force-
isotropic configuration is
\[ (k_1, k_2, k_3, k_4, \theta_1, \theta_2, \theta_3, \theta_4) = 
(33/128, 1/2, 2/3, 1/2, \pi/2, -\pi/6, \pi/3, \pi/3) \]
(38)
considering a unitary proximal phalanx length. In order to
simplify the example, these parameters have been chosen so
that they correspond to the same force-isotropic finger defined
by the following geometric parameters:
\[ (l_1, l_2, \psi, a, b, c) = (1, 2/3, \pi/2, 6/5, 6/5, 1/3) \]
(39)
These parameters were obtained using the procedure described
in [13]. In this case, one can obtain geometric parameters that
lead to a force-isotropic design, e.g.,
\[ (e_1, e_2, c_1, c_2, r_1, r_2, a_1, a_2) = 
(1/2, 1/2, 1/3, 1/3, 1, 1, 17/20, 17/20) \]
(40)
2) Global optimization: a fundamental drawback of the
previous method is that it requires a priori knowledge of the
grasped object which might not be available. Furthermore,
force-isotropy for typical objects might not be mandatory
or difficult to obtain with reasonable geometric parameters.
Alternatively, global performance indices can be defined using
the definition of the stage force transmission ratio given in
eq. (6). The contact forces generated by the actuation force
can be easily computed by combining the results of [9] and
Section II. Hence, performance indices based on the output forces can be defined. One has

\[
\begin{bmatrix}
  f_1 \\
f_2 \\
f_3 \\
f_4
\end{bmatrix}
= \begin{bmatrix}
  J_L^T T_L^T d_L \\
  0 \\
  0 \\
  J_R^T T_R^T d_R
\end{bmatrix}
\]

\[
\times \begin{bmatrix}
  b_2/c \\
b_1/c \\
-(\sin \alpha_1)/c
\end{bmatrix}
\begin{bmatrix}
  F_a \\
  T^s
\end{bmatrix}
\]

which is the equation defining the contact forces as functions of the input actuation force \( F_a \). Matrices \( J_L \) and \( J_R \) are the Jacobian matrices of the grasp [9] for, respectively, the left and right fingers. Similarly, \( T_L \) and \( T_R \) are the kinematic Transmission matrices of the left and right fingers [9]. Vectors \( d_L \) and \( d_R \) are defined as

\[
d_L = \begin{bmatrix} d_1 \\ T_L^s \end{bmatrix}, \quad d_R = \begin{bmatrix} d_2 \\ T_R^s \end{bmatrix}.
\]

The torques of the spring of the left and right fingers are respectively denoted \( T_L^s \) and \( T_R^s \). An index of interest could thereupon be defined as the ratio of the total grasp force to the actuation force, namely

\[
I = \frac{\int_W \left( \sum_{i=1}^{4} f_i \right) d\theta_a}{\int_W F_a d\theta_a}
\]

(43)

Note that the workspace \( W \) of the gripper is defined as the set of all contact configurations, described by \( \theta_a \), \( [k_1 \; k_2 \; k_3 \; k_4 \; \theta_1 \; \theta_2 \; \theta_3 \; \theta_4]^T \), where all contact forces are positive. This index is the average mechanical efficiency of the gripper as a whole. It has been shown [18] that a value close to one is of uttermost importance for instance in case of a manually driven surgical gripper. Indeed, this property allows to perceive the pulse of an artery and the stiffness differences between diseased and healthy tissue by the handler of the instrument without complex electronic feedback [18]. Surgical grippers can take advantage of the shape-adaptation capability provided using underactuation, which allows them to perform enveloping grasps automatically and therefore, distribute the contact forces along a larger surface than conventional grippers. An analytical expression of the index defined in eq. (43) can be computed, provided that a few hypotheses on the gripper design and contact locations are made, namely

1) both fingers are identical, i.e., \( l_1 = l_3 \) and \( l_2 = l_4 \),
2) the springs are neglected,
3) the contact points are located at mid-phalaxn, i.e., \( k_1 = k_3 = l_1/2 \) and \( k_2 = k_4 = l_2/2 \),
4) the range of motion for the angles are \( \pi/4 < \theta_1, \theta_3 < 3\pi/4 \) and \( 0 < \theta_2, \theta_1 < \pi/2 \).

If the latter hypotheses are satisfied, one obtains

\[
I = \int_W I^* d\theta
\]

(44)

where \( \theta = [\theta_1 \; \theta_2 \; \theta_3 \; \theta_4]^T \) and with

\[
I^* = \frac{32}{l_2 c \pi^4} [A + B]
\]

(45)

with

\[
A = b_2 d_1 \left( l_2 - 2h_1 \cos \theta_2 + h_1 \right) / (h_1 + l_1)
\]

(46)

\[
B = b_1 d_2 \left( l_2 - 2h_2 \cos \theta_4 + h_2 \right) / (h_2 + l_1)
\]

(47)

For instance, if the geometric parameter set of Table I is chosen for the fingers, one can optimize the differential stage accordingly. If further parameters are chosen, e.g.,

\[
e_1 = e_2 = c_1 = c_2 = 1/2
\]

(48)

and \( r_1 = r_2 = 1 \), the value of \( I \) can be computed as a function of \( a^* = a_1 = a_2 \). The value of the index as a function of \( a^* \) is illustrated in Fig. 14. It can be seen in this Figure that the optimal value of \( a^* \) is approximatively 1.40 with respect to the index. If this length is chosen, the average force applied by the user (e.g. a surgeon) in the gripper’s previously defined workspace is the same as the force exerted on the object seized. Hence, the total squeezing force applied to the object is globally neither amplified, nor diminished, allowing the user to finely control the grasping force.

![Fig. 14. Index I as a function of a*.

Table I

<table>
<thead>
<tr>
<th>Geometric parameters</th>
</tr>
</thead>
<tbody>
<tr>
<td>( l_1 )</td>
</tr>
<tr>
<td>1</td>
</tr>
</tbody>
</table>

An example of underactuated gripper using the design of Fig. 12 is presented in [19] and shown in Fig. 15. The gripper is constituted of compliant hinges due to its small size (less than 10 mm) and is machined from a single block of NiTiNol, a nickel-titanium bio-compatible shape memory alloy with remarkable elastic characteristics. However, it should be noted that this preliminary prototype has not been designed using the methodology discussed above, but only considering fingers individually.
B. Multiple pulley routing

The two combinations of movable pulleys that can be used to drive four outputs with one input are illustrated in Fig. 16. The first routing is serial according to the definition used in Section III-A, meaning that one output of each stage is directly used to generate one of the final output forces while the other is propagated to the upper stage of the transmission tree. The second routing is fully symmetrical, again using the definition given in Section III-A, namely all outputs of each layer of the transmission tree are used to propagate the input except for the last layer whose outputs are the final force outputs.

The cables used in the serial routing are indicated by solid lines (between the pulleys), the output forces of this layout are $F_i^a$ with $i = 2, \ldots, 5$. In the symmetrical layout, the cables are indicated by dashed lines and the associated output forces are $F_i^a$ with $i = 1, \ldots, 4$. In order to simplify the expressions of the associated transmission matrices, it is assumed that the distance between two consecutive force outputs are identical and the same as the distance between two consecutive axes of the movable pulleys. Nevertheless, the method presented is general and can be easily extended to more general cases.

1) Serial routing: Using the results of Section III-A, the global transmission matrix of the system with this routing is

$$
\begin{align*}
\begin{pmatrix}
\frac{r_2}{c_2} & \sin \alpha_4 \\
r_3 r_2 & -r_3 \sin \alpha_3 \\
r_4 r_3 c_2 & -r_4 r_3 \sin \alpha_3 \\
r_5 r_4 r_3 c_2 & -r_5 r_4 r_3 \sin \alpha_3 \\
r_6 r_5 r_4 r_3 c_2 & -r_6 r_5 r_4 r_3 \sin \alpha_3 \\
n + 1 & 0
\end{pmatrix}
\end{align*}
$$

with

$$
\begin{align*}
\alpha_7 = & \frac{\pi}{2} - 2 \arctan \left( \frac{-(h_i - h_{i+1}) + \sqrt{(2d)^2 - r_3^2 + (h_i - h_{i+1})^2}}{2d + r_3} \right) \\
\alpha_8 = & \frac{\pi}{2} - 2 \arctan \left( \frac{-h_4 + \sqrt{r_4^2 - r_3^2 + h_4^2}}{d + r_4} \right)
\end{align*}
$$

for $i = 2, 3, 4$. If the springs are neglected, relatively simple expressions of the output forces can be obtained, namely

$$
F_i^a = \frac{r_2}{c_2} F_a, \quad F_i^a = \frac{r_3 r_2}{c_3 c_2} F_a, \\
F_i^a = \frac{r_4 r_3 r_2}{c_4 c_3 c_2} F_a, \quad F_i^a = \frac{r_5 r_4 r_3 r_2}{c_5 c_4 c_3 c_2} F_a.
$$

It is readily observed that the force transmission ratio of one stage also appears in the force transmission ratio of all the next stages. This characteristic is typical of a serial transmission tree. The main drawback of the serial routing arises from this coefficient propagation. Since generally $c_i > r_i$, this coefficient is smaller than one and hence the output forces tend to decrease when the number of stages increases. Similarly, in underactuated fingers, the distal phalanx is usually much weaker than the proximal phalanx. Therefore force-isotropy cannot be achieved with such a layout. However, this characteristic can be of use if a thumb-based layout is desired. Indeed, if the cable connected to the thumb corresponds to $F_2^a$, the coefficients $r_i/c_i$ for $i = 3, 4$ can be adjusted to ensure

$$
F_2^a = \sum_{i=3}^5 F_i^a \Rightarrow \frac{r_3}{c_3} \left( 1 + \frac{2 r_4}{c_4} \right) = 1
$$

Hence, the fingers opposing the thumb balance the force generated by the latter. However, this characteristic is generally only local since the coefficients $c_i$ are functions of the kinematic state of the outputs.
2) Symmetrical routing: The expression of the global transmission matrix if a symmetrical routing is chosen is

\[
\begin{bmatrix}
  r_1 r_2 & r_1 \sin \alpha_4 & \sin \alpha_2 & 0 \\
  c_1 c_2 & c_1 c_2 & c_1 & 0 \\
  r_1 r_2 & r_1 \sin \alpha_4 & -c_1 & 0 \\
  c_1 c_2 & c_1 c_2 & -c_1 & 0 \\
  r_3 r_2 & r_3 \sin \alpha_3 & 0 & \sin \alpha_5 \\
  c_3 c_2 & c_3 c_2 & c_3 & 0 \\
  r_3 r_2 & r_3 \sin \alpha_3 & 0 & -\sin \alpha_5 \\
  c_3 c_2 & c_3 c_2 & -c_3 & 0 
\end{bmatrix}
\]  

(55)

where the notation used is the same as in the previous case. These angles \( \alpha_i \) can again be expressed as functions of the pulley positions \( h_i \) but since the routing is different, the expressions slightly differ, i.e.,

\[
\begin{align*}
\alpha_{2i-1} &= \frac{\pi}{2} + 2 \arctan \left( \frac{h_i + \sqrt{r_1^2 - r_3^2 + h_i^2}}{r_1 + r_3} \right) \\
\alpha_{2i} &= \frac{\pi}{2} - 2 \arctan \left( \frac{-(h_i - h_{i+1}) + \sqrt{(2d)^2 - r_1^2 + (h_i - h_{i+1})^2}}{2d + r_1} \right) 
\end{align*}
\]

(56)

for \( i = 1, 3 \), and

\[
\begin{align*}
\alpha_3 &= \frac{\pi}{2} + 2 \arctan \left( \frac{-(h_2 - h_1) + \sqrt{(2d)^2 - r_2^2 + (h_2 - h_1)^2}}{4d + r_2} \right) \\
\alpha_4 &= \frac{\pi}{2} - 2 \arctan \left( \frac{-(h_2 - h_3) + \sqrt{(2d)^2 - r_3^2 + (h_2 - h_3)^2}}{2d + r_2} \right).
\end{align*}
\]

(57)

If the springs are neglected, the expressions of the output forces are

\[
\begin{align*}
F_a &= \frac{r_1 r_2}{c_1 c_2} F_a, & F_2 &= \frac{r_1 r_2}{c_1 c_2} F_a, \\
F_3 &= \frac{r_3 r_2}{c_3 c_2} F_a, & F_4 &= \frac{r_2 r_3}{c_2 c_3} F_3.
\end{align*}
\]

(58)

In this case of routing, it can be observed that the force transmission ratio of each output is the product of the force transmissions of both the stage located immediately before the output and the input stage. Hence, in this case, force-isotropy is possible and easily achievable. Namely, the transmission is force-isotropic if and only if

\[
\frac{r_1}{c_1} = \frac{r_3}{c_3}
\]

(59)

However, this property is again usually only local. Since force-isotropy can be achieved with such a layout, it can readily be applied to non-anthropomorphic, spherical grippers.

An example of a prosthetic hand built with Fused Deposition Modeling (FDM) rapid-prototyping technology and using a multiple pulley routing is presented in Fig. 17, from [20]. A single wire is used to drive five fingers with one thumb opposing the four other fingers. The four fingers are driven through a fully symmetrical transmission tree while another pulley is added to the output to drive the thumb. The prosthesis therefore provides 15 DOF and self-adaptability to the various shapes of seized objects with only one actuator.

In this paper, a methodology was proposed for the analysis of the force capabilities of common differential mechanisms used to extend the principle of underactuation from the fingers to the hand itself. In the first part of the paper, several differential elements, most noticeably used in robotic hands to provide underactuation between a certain number of fingers, were presented. A simple matrix formulation was developed to obtain the relationship between the actuation and output forces of the devices. Then, a mathematical method to obtain the output force capabilities of connected differential mechanisms was presented. The application of the method to the analysis of strictly serial and fully symmetrical transmission trees was presented. Two examples were then analyzed using the technique presented in the paper: first an underactuated gripper and second a multiple pulley routing. The mathematical expressions of the transmission tree characterizing the underactuated system considered are fundamentally different from the Transmission matrix arising when one considers underactuation in the fingers instead of between them. To the best of the authors’ knowledge, the force capabilities of connected differential mechanisms are mathematically studied here for the first time.

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**REFERENCES**


